

Measurable regularity properties of infinite-dimensional Lie groups

Helge Glöckner

Abstract

Let G be a Banach-Lie group with Lie algebra \mathfrak{g} , and $p \in [1, \infty]$. Then the space $AC_{L^p}([0, 1], \mathfrak{g})$ of absolutely continuous functions $\gamma: [0, 1] \rightarrow \mathfrak{g}$ with $\gamma' \in L^p([0, 1], \mathfrak{g})$ is a Banach-Lie algebra. Let $AC_{L^p}([0, 1], G)_0 = \langle \exp_G \circ \gamma: \gamma \in AC_{L^p}([0, 1], \mathfrak{g}) \rangle$ be the integral subgroup of $C([0, 1], G)$ with Lie algebra $AC_{L^p}([0, 1], \mathfrak{g})$. We show that each $\gamma \in L^p([0, 1], \mathfrak{g})$ has a left evolution $\text{Evol}(\gamma) \in AC_{L^p}([0, 1], G)_0$, and that the map $\text{Evol}: L^p([0, 1], \mathfrak{g}) \rightarrow AC_{L^p}([0, 1], G)_0$ is smooth. Similar results are obtained for important classes of Fréchet-Lie groups and more general Lie groups, notably for diffeomorphism groups of paracompact finite-dimensional smooth manifolds and gauge groups of principal bundles with Banach structure groups. The measurable regularity properties considered imply validity of the Trotter product formula and the commutator formula.

Classification: 22E65 (primary); 32A12, 34G10, 46G20, 46H05, 58B10.

Key words: Infinite-dimensional Lie group, Banach-Lie group, Fréchet-Lie group, regular Lie group, regularity, logarithmic derivative, product integral, evolution, initial value problem, parameter dependence, measurable map, current group, loop group, gauge group, diffeomorphism group, projective limit, direct limit, inductive limit, direct sum, weak direct product, extension, test function group, locally m-convex algebra, continuous inverse algebra, commutator formula, Trotter formula, Lebesgue space, regulated function, absolute continuity, Carathéodory solution, measurable right-hand-side, control theory

Introduction and statement of the main results

To enable proofs for fundamental Lie theoretic facts in infinite dimensions, John Milnor [52] introduced the concept of regularity for infinite-dimensional Lie groups (compare also [58], [59] for related earlier work). Let G be a Lie group modelled on a locally convex space E , with identity element e , tangent bundle $T(G)$ and Lie algebra $\mathfrak{g} := L(G) := T_e(G) \cong E$. Given

$g, h \in G$ and $v \in T_h(G)$, let $\lambda_g: G \rightarrow G$, $x \mapsto gx$ be left translation and $g.v := T_h(\lambda_g)(v) \in T_{gh}(G)$. Thus $h^{-1}.v \in T_e(G) = \mathfrak{g}$. If $\gamma: [0, 1] \rightarrow \mathfrak{g}$ is a continuous map, then there exists at most one C^1 -map $\eta: [0, 1] \rightarrow G$ with

$$\eta'(t) = \eta(t).\gamma(t) \quad \text{for all } t \in [0, 1], \text{ and } \eta(0) = 1. \quad (1)$$

If such an η exists, it is called the *evolution of* γ , and denoted by $\text{Evol}(\gamma) := \eta$. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. The Lie group G is called *C^k -regular* if each $\gamma \in C^k([0, 1], \mathfrak{g})$ admits an evolution $\text{Evol}(\gamma)$ and the map $\text{Evol}: C^k([0, 1], \mathfrak{g}) \rightarrow C^{k+1}([0, 1], G)$ is smooth with respect to the natural Lie group structure on the mapping group $C^{k+1}([0, 1], G)$ (cf. [30], [34], and [54]). Then C^k -regularity implies C^ℓ -regularity for all $\ell \in \mathbb{N}_0 \cup \{\infty\}$ with $\ell \geq k$ (cf. [34]). The C^∞ -regular Lie groups are simply called *regular* (cf. [34], [52], and [54], where also applications of regularity are described). For the purposes of representation theory, the strongest notion, C^0 -regularity, is particularly useful [55]. Recently, it also proved valuable to consider weakened topologies on $C^0([0, 1], \mathfrak{g})$ (like the L^1 -topology) [34].

Again for the purposes of representation theory, it would be useful to have even stronger regularity properties available. And also from the point of view of control theory, it is natural to allow more general functions γ on the right hand side of (1), e.g. step functions (with steps when a control is switched on or off). The current article is devoted to such generalizations. We mention that initial value problems of the form (1), for G a finite-dimensional Lie group, are a familiar topic in Geometric Control Theory.¹ See, e.g., [42, Chapter 12] for optimal control problems in this context, when the controls are furnished by left-invariant vector fields on G . For general aspects of control theory on finite-dimensional spaces with measurable right hand sides (irrespective of Lie groups) and the corresponding initial value problems, see [68].

If E is a Fréchet space (resp., a Banach space) and $p \in [1, \infty]$, let $L^p([0, 1], E)$ be the space of all (equivalence classes of) measurable functions $\gamma: [0, 1] \rightarrow E$ with separable image such that $q \circ \gamma$ is in the Lebesgue space $\mathcal{L}^p([0, 1], \mathbb{R})$ for each continuous seminorm q on E (cf. [41] and [62] if E is Banach). We let $AC_{L^p}([0, 1], E)$ be the space of all functions $\eta: [0, 1] \rightarrow E$ of the form

$$\eta(t) = v + \int_0^t \gamma(s) d\lambda_1(s)$$

¹In this context, γ in (1) is usually parametrized by certain control functions.

with $v \in E$ and $[\gamma] \in L^p([0, 1], E)$ (where λ_1 is Lebesgue-Borel measure on \mathbb{R}). Then $\eta'(t) = \gamma(t)$ for λ_1 -almost all $t \in [0, 1]$ (see Lemma 1.28; cf. [62] if E is Banach). The spaces $L^p([0, 1], E)$ and $AC_{L^p}([0, 1], E)$ are Fréchet spaces (resp., Banach spaces) in a natural way (see, e.g., Lemma 1.19). Testing in local charts, one can also speak of AC_{L^p} -maps to a manifold M modelled on a Fréchet space (cf. Definition 3.20). We show that $AC_{L^p}([0, 1], G)$ is a Fréchet-Lie group (resp., Banach-Lie group), for each Fréchet-Lie group (resp., Banach-Lie group) G (Proposition 4.2). Let G be a Fréchet-Lie group, with Lie algebra \mathfrak{g} . Let $p \in [1, \infty]$. We say that G is L^p -regular if each $[\gamma] \in L^p([0, 1], \mathfrak{g})$ has a (necessarily unique) left evolution $\text{Evol}([\gamma]) := \eta \in AC_{L^p}([0, 1], G)$ such that

$$\eta(0) = e \quad \text{and} \quad \eta'(t) = \eta(t) \cdot \gamma(t) \quad \lambda_1\text{-almost everywhere,}$$

and the map $\text{Evol}: L^p([0, 1], \mathfrak{g}) \rightarrow AC_{L^p}([0, 1], G)$ is smooth. Let $L_{rc}^\infty([0, 1], \mathfrak{g})$ be the space of all elements of $L^\infty([0, 1], \mathfrak{g})$ having a representative γ with relatively compact image. Then $L_{rc}^\infty([0, 1], \mathfrak{g})$ is a closed vector subspace of $L^\infty([0, 1], \mathfrak{g})$. A function $\gamma: [0, 1] \rightarrow \mathfrak{g}$ is called *regulated* if it is a uniform limit of a sequence of E -valued step functions. Then the space $R([0, 1], \mathfrak{g})$ of equivalence classes of regulated maps is a closed vector subspace of $L_{rc}^\infty([0, 1], \mathfrak{g})$ (and hence also of $L^\infty([0, 1], \mathfrak{g})$). If $L^\infty([0, 1], \mathfrak{g})$ is replaced with $L_{rc}^\infty([0, 1], \mathfrak{g})$ and $R([0, 1], \mathfrak{g})$ in the above definition, then G is called L_{rc}^∞ -regular and R -regular (or *regulated regular*), respectively.

Theorem A. *For each Fréchet-Lie group G and each $p \in [1, \infty]$, we have the following implications:*

$$G \text{ is } L^p\text{-regular} \Rightarrow G \text{ is } L^q\text{-regular for all } q \in [1, \infty] \text{ such that } q \geq p;$$

$$G \text{ is } L^\infty\text{-regular} \Rightarrow G \text{ is } L_{rc}^\infty\text{-regular} \Rightarrow G \text{ is } R\text{-regular};$$

$$G \text{ is } R\text{-regular} \Rightarrow G \text{ is } C^0\text{-regular}.$$

We say that a locally convex space E is *integral complete* if each continuous curve $\gamma: [0, 1] \rightarrow E$ has a weak integral $\int_a^b \gamma(t) dt \in E$. It is known that E is integral complete if and only if E has the *metric convex compactness property* (metric CCP) in the sense that the closed convex hull

$$\overline{\text{conv}(K)} \subseteq E$$

is compact for each compact metrizable subset $K \subseteq E$ (see [71]). The following implications are known for a locally convex space E :

$$\begin{aligned} E \text{ is complete} &\Rightarrow E \text{ is quasi-complete} \Rightarrow E \text{ is sequentially complete} \\ &\Rightarrow E \text{ is integral complete;} \end{aligned}$$

moreover, none of these implications are equivalences [70]. If we try to strengthen the concept of C^0 -regularity for a Lie group G modelled on a locally convex space E , then E has to be integral complete (as $E \cong L(G)$ is integral complete for each C^0 -regular Lie group G [34]). We mention that Hausdorff locally convex spaces $L_{rc}^\infty([0, 1], E)$ can be defined for E an arbitrary locally convex space, formed by equivalence classes of all measurable functions $\gamma: [0, 1] \rightarrow E$ such that the closure $\overline{\gamma([0, 1])}$ of the image is compact and metrizable (see [24]). If E is integral complete, then it is possible to define spaces $AC_{L_{rc}^\infty}([0, 1], E)$ (see Section 3), giving rise to Lie groups $AC_{L_{rc}^\infty}([0, 1], G)$ and notions of L_{rc}^∞ -regularity and R -regularity for arbitrary Lie groups G modelled on integral complete locally convex spaces (see Sections 4 and 5).

We say that a locally convex space E has the *Fréchet exhaustion property* (FEP) if every closed vector subspace $S \subseteq E$ having a dense countable subset can be written as the union

$$S = \bigcup_{n \in \mathbb{N}} F_n$$

of an ascending sequence $F_1 \subseteq F_2 \subseteq \dots$ of vector subspaces $F_n \subseteq E$ which are Fréchet spaces in the induced topology (see Definition 1.38). For every locally convex space E with the (FEP), we are able to give a sense to $L^p([0, 1], E)$ with $p \in [1, \infty]$ (see 1.40). The class of (FEP)-spaces subsumes all Fréchet spaces and strict (LF)-spaces $E = \varinjlim E_n$. Moreover, the space $\mathcal{X}_c(M)$ of compactly supported smooth vector fields on a paracompact finite-dimensional smooth manifold M is an (FEP)-space (see Lemma 1.41 for these assertions). The latter is the modelling space for the Lie group

$$\text{Diff}_c(M)$$

of all diffeomorphisms $\phi: M \rightarrow M$ such that, for some compact set $K \subseteq M$, we have $\phi(x) = x$ for all $x \in M \setminus K$ (cf. [51], [22], or [63]).

Two main results are devoted to measurable regularity properties of

Banach-Lie groups and diffeomorphism groups.² For M a paracompact finite-dimensional smooth manifold and $K \subseteq M$ a compact set, let $\text{Diff}_K(M)$ be the group of all smooth diffeomorphisms $\phi: M \rightarrow M$ such that $\phi(x) = x$ for all $x \in M \setminus K$. We obtain the following result:

Theorem B. *For each paracompact finite-dimensional smooth manifold M , the Lie group $\text{Diff}_c(M)$ is L^1 -regular. Also $\text{Diff}_K(M)$ is L^1 -regular, for each compact subset $K \subseteq M$.*

We first prove Theorem B for the instructive cases of $\text{Diff}_K(\mathbb{R}^n)$, $\text{Diff}_c(\mathbb{R}^n)$ and $\text{Diff}(\mathbb{S}_1)$, before turning to general M . We also prove:

Theorem C. *Every Banach-Lie group is L^1 -regular.*

Using a projective limit argument, we deduce that also some Fréchet-Lie groups are L^1 -regular. For example, the unit group A^\times is L^1 -regular for each continuous inverse algebra A which is a Fréchet space and locally m -convex (in the sense of [50]): see Proposition 7.15. Recall that a continuous inverse algebra is a unital associative real (or complex) algebra, endowed with a locally convex vector topology for which the unit group $A^\times \subseteq A$ is open and both the inversion map $A^\times \rightarrow A$, $a \mapsto a^{-1}$ and the algebra multiplication $A \times A \rightarrow A$ are continuous; then A^\times is a Lie group (see [21] and the references therein). Similarly, we find that the mapping group

$$C_K^\infty(M, H) := \{\gamma \in C^\infty(M, H) : \gamma|_{M \setminus K} = e\}$$

is L^1 -regular for each finite-dimensional smooth manifold M , Banach-Lie group H and compact set $K \subseteq M$ (Proposition 7.11).

We then turn to measurable regularity properties of ascending unions of Lie groups, and related topics. Notably, we find that the weak direct product Lie group $\bigoplus_{j \in J} H_j$ (as introduced in [24]) is L^1 -regular for each family of L^1 -regular Lie groups H_j modelled on sequentially complete (FEP)-spaces (Proposition 8.2). Together with a result on the inheritance of measurable regularity properties by certain Lie subgroups (Proposition 5.27), this entails:

Theorem D. *The test function group $C_c^k(M, H)$ is L^1 -regular, for each paracompact finite-dimensional smooth manifold M , Banach-Lie group H and*

²The L_{rc}^∞ -regularity of Banach-Lie groups was first announced in [24] (without proof, and using different terminology); the L_{rc}^∞ -regularity of $\text{Diff}_c(M)$ was conjectured there.

$k \in \mathbb{N}_0 \cup \{\infty\}$.

Here $C_c^k(M, H)$ is the Lie group of all C^k -maps $\gamma: M \rightarrow H$ such that $\gamma^{-1}(H \setminus \{e\}) \subseteq M$ is relatively compact; it is modelled on the locally convex direct limit $C_c^k(M, L(H)) = \varinjlim C_K^k(M, L(H))$ (see, e.g., [20] for the Lie group structure in the main case that M is σ -compact). The conclusion of Theorem D remains valid for Lie groups of compactly supported gauge transformations of principal bundles with Banach structure groups (see Corollary 8.3).

We also have a result ensuring measurable regularity properties for ascending unions of Banach-Lie groups under suitable hypotheses (Proposition 8.10). It entails:

Theorem E. *Let $G_1 \subseteq G_2 \subseteq \dots$ be finite-dimensional Lie groups, such that the inclusion maps $G_n \rightarrow G_{n+1}$ are smooth group homomorphisms for all $n \in \mathbb{N}$. Then the direct limit Lie group $\varinjlim G_n = \bigcup_{n \in \mathbb{N}} G_n$ (as in [23], [27]) is L^1 -regular.*

In particular, $G = \varinjlim G_n$ is always C^0 -regular. So far, it was only known that $G = \varinjlim G_n$ is $\overline{C^1}$ -regular [27].³

Proposition 8.10 also implies (see Corollary 8.20):

Theorem F. *For each compact real analytic manifold M and each Banach-Lie group H , the Lie group $C^\omega(M, H)$ of all real analytic H -valued maps on M is L_{rc}^∞ -regular.*

The C^0 -regularity of $C^\omega(M, H)$ is already stated in [16].

All measurable regularity properties we consider are extension properties:

Theorem G. *Consider an extension*

$$\{1\} \rightarrow N \xrightarrow{j} G \rightarrow Q \xrightarrow{q} \{1\}$$

of Lie groups modelled on integral complete locally convex spaces, such that q admits smooth local sections. If both N and Q are L_{rc}^∞ -regular (resp., R -regular), then also G is L_{rc}^∞ -regular (resp., R -regular). If $p \in [0, \infty]$ and both

³In the special situation of [14] (which, in particular, entails that \exp_G is a local diffeomorphism at 0) C^0 -regularity was already available.

N and Q are L^p -regular Fréchet-Lie groups, then also G is an L^p -regular Fréchet-Lie group. If N and Q are L^p -regular Lie groups modelled on sequentially complete (FEP)-spaces, then also G is modelled on a sequentially complete (FEP)-space and L^p -regular.

Even the weakest measurable regularity established here, R -regularity, has remarkable consequences. Let G be a Lie group modelled on a locally convex space such that G has a smooth exponential function⁴ $\exp_G: \mathfrak{g} \rightarrow G$ on $\mathfrak{g} = L(G)$. Following [55], G is said to have the *Trotter property* if, for all $v, w \in \mathfrak{g}$,

$$(\exp_G(tv/n) \exp_G(tw/n))^n$$

converges to $\exp_G(t(v+w))$ as $n \rightarrow \infty$, uniformly for t in compact subsets of \mathbb{R} . We say that G has the *strong Trotter property* if even⁵

$$(\gamma(t/n))^n \rightarrow \exp_G(t\gamma'(0)) \quad \text{as } n \rightarrow \infty, \quad (2)$$

uniformly for t in compact subsets of $[0, \infty[$, for each C^1 -curve $\gamma: [0, 1] \rightarrow G$ such that $\gamma(0) = e$. If, for all $v, w \in \mathfrak{g}$,

$$(\exp_G(\sqrt{t}/n) \exp_G(\sqrt{t}/n) \exp_G(-\sqrt{t}/n) \exp_G(-\sqrt{t}/n))^{n^2} \rightarrow \exp_G(t[v, w])$$

as $n \rightarrow \infty$, uniformly in t in compact subsets of $[0, \infty[$, then we say that G has the *commutator property*. We say that G has the *strong commutator property* if

$$(\gamma(\sqrt{t}/n) \eta(\sqrt{t}/n) \gamma(\sqrt{t}/n)^{-1} \eta(\sqrt{t}/n)^{-1})^{n^2} \rightarrow \exp_G(t[\gamma'(0), \eta'(0)]) \text{ as } n \rightarrow \infty,$$

uniformly for t in compact subsets of $[0, \infty[$, for all C^1 -curves $\gamma, \eta: [0, 1] \rightarrow G$ such that $\gamma(0) = \eta(0) = e$. Both the Trotter property and the commutator property are useful in representation theory (see [55] and ongoing work by K.-H. Neeb). We already explained that the strong Trotter property implies the Trotter property. Likewise, the strong commutator property implies the commutator property. We show:

Theorem H. *Let G be a Lie group modelled on a locally convex space. If G*

⁴This ensures that $\mathbb{R} \rightarrow G, t \mapsto \exp_G(tv)$ is a smooth one-parameter group of G for each $v \in \mathfrak{g}$ with $\frac{d}{dt}|_{t=0} \exp_G(tv) = v$, and that every smooth one-parameter group of G is of this form.

⁵This implies the Trotter property, as we can take $\gamma(t) := \exp_G(tv) \exp_G(tw)$.

has the strong Trotter property, then G has the strong commutator property.

Theorem I. *Let G be a Lie group modelled on an integral complete locally convex space. If G is R -regular, then G has the strong Trotter property (and hence also the strong commutator property).*

We mention that the notion of R -regularity provides a link to the original notion of regularity (called μ -regularity in [54]) in the works by Omori and collaborators (see [58] and [59]), which was based on the convergence of certain “product integrals.” In the special case of Fréchet-Lie groups, the assertion on the strong Trotter property in Theorem G for R -regular Lie groups is a counterpart of the corresponding result for μ -regular Fréchet-Lie groups in [59, Lemma 1.1].

Combining Theorems D and I, we see that the test function group $C_c^k(M, H)$ has the strong Trotter property for each paracompact finite-dimensional smooth manifold M and Banach-Lie group H . Generalizing the case of $C_c^\infty(M, H)$, also the gauge group $\text{Gau}(P)$ of a principal bundle $P \rightarrow M$ with structure group H (with $\text{Gau}_c(P)$ as an open Lie subgroup) is L^1 -regular and hence has the strong Trotter property, for each paracompact finite-dimensional smooth manifold M and Banach-Lie group H . Also $\text{Diff}(M)$ (with $\text{Diff}_c(M)$ as an open Lie subgroup) is L^1 -regular and hence has the strong Trotter property. Now the full automorphism group $\text{Aut}(P)$ is a Lie group extension

$$\{1\} \rightarrow \text{Gau}(P) \rightarrow \text{Aut}(P) \rightarrow \text{Diff}(M)_P \rightarrow \{1\}$$

for a suitable open subgroup $\text{Diff}(M)_P$ of $\text{Diff}(M)$ (cf. [65] for the essential special case that M is σ -compact). Since being L^1 -regular is an extension property, we deduce:

Theorem J. *Let $P \rightarrow M$ be a smooth principal bundle over a paracompact finite-dimensional smooth manifold M whose structure group is a Banach-Lie group. Then $\text{Aut}(P)$ is L^1 -regular and hence $\text{Aut}(P)$ has the strong Trotter property and the strong commutator property.*

Our proof of Theorem I shows that the convergence in (2) is even uniform for γ in compact sets. This implies:

Theorem K. *If a Lie group H is R -regular, then $C_K(X, H)$ and $C_c(X, H)$*

have the strong Trotter property, for every locally compact topological space X and compact subset $K \subseteq X$. If, moreover, X is paracompact, then also $C_c(X, H)$ has the strong Trotter property.

Here $C_K(X, H) := \{\gamma \in C(X, H) : \gamma|_{X \setminus K} = e\}$ is endowed with its natural Lie group structure (see, e.g., [20]); likewise for $C_c(X, H) = \bigcup_K C_K(X, H)$ (cf. [20] for the essential case when X is σ -compact).

Note that the Lebesgue spaces $L^p([0, 1], E)$, $L_{rc}^\infty([0, 1], E)$ and $R([0, 1], E)$ we consider only serve as a tool to define strengthened regularity properties, where they appear as the domains of certain evolution maps. For this purpose, properties like completeness of the spaces (which might fail unless we assume that E is a Fréchet space) are irrelevant. Rather, it is important that we have good results on continuity and differentiability properties for mappings between such spaces or families of such. Results of this type do not seem available if, instead, one would define vector-valued L^p -spaces as completions of tensor products $L^p[0, 1] \otimes E$ with respect to suitable tensor norms (which might look more natural from the point of view of linear functional analysis). Compare [18] for another viable type of vector-valued L^p -maps based on Suslin-measurability.

For previous work concerning differential equations in finite-dimensional (or Banach) spaces with measurable right hand sides, see e.g. [44], [62], [67], [68] and the references therein.

Structure of the article. After a preparatory section with selected material on Lebesgue spaces and infinite-dimensional calculus (Section 1), we study differentiability properties of mappings like

$$\tilde{f}: C([a, b], V) \times L^p([a, b], E_2) \rightarrow L^p([a, b], F), \quad (\eta, [\gamma]) \mapsto [f \circ (\eta, \gamma)],$$

e.g. if E_2 and F are Fréchet spaces, V is an open subset of a locally convex space E_1 and

$$f: V \times E_2 \rightarrow F$$

a smooth map which is linear in its second argument (Section 2). If E is a Fréchet space, then we can consider each of the spaces

$$L^p([a, b], E), \quad L_{rc}^\infty([a, b], E) \quad \text{and} \quad R([a, b], E)$$

as a vector subspace $\mathcal{E}([a, b], E)$ of $L^1([a, b], E)$. The assignment is functorial

both in E and in $[a, b]$; e.g., we have a continuous linear map

$$L^p([a, b], \lambda): L^p([a, b], E_1) \rightarrow L^p([a, b], E_2), \quad [\gamma] \mapsto [\lambda \circ \gamma]$$

for each continuous linear map $\lambda: E_1 \rightarrow E_2$ between Fréchet spaces. And we can pull back functions along an affine-linear map $f: [c, d] \rightarrow [a, b]$:

$$L^p(f, E): L^p([a, b], E) \rightarrow L^p([c, d], E), \quad [\gamma] \mapsto [\gamma \circ f].$$

We therefore speak of a *bifunctor* \mathcal{E} on Fréchet spaces. Using such a bifunctor, we call a function

$$\eta: [a, b] \rightarrow E$$

\mathcal{E} -absolutely continuous (and write $\eta \in AC_{\mathcal{E}}([a, b], E)$) if

$$\eta(t) = \eta(a) + \int_a^t \gamma(s) ds$$

for some $[\gamma] \in \mathcal{E}([a, b], E)$. The theory of vector-valued absolutely continuous functions is developed in Section 3. Notably, we study differentiability properties of non-linear mappings on spaces of absolutely continuous functions and describe conditions ensuring that absolutely continuous functions to manifolds can be defined. This enables a Lie group structure on

$$AC_{\mathcal{E}}([0, 1], G)$$

to be constructed for suitable bifunctors \mathcal{E} and Lie groups G (Section 4), which are then used to define and study \mathcal{E} -regular Lie groups (Section 5). To this end, certain axioms and properties need to be imposed on the bifunctors under consideration. Thus, we shall encounter the Locality Axiom, the Pushforward Axioms, the Subdivision Property, and the requirement that smooth functions act smoothly on $AC_{\mathcal{E}}$. We shall see that all of these axioms and requirements are satisfied by L^p , L_{rc}^{∞} and R .⁶ Thus \mathcal{E} -regularity provides a common roof (and uniform proofs) for the concepts of L^p -regular, L_{rc}^{∞} -regular and R -regular Lie groups. As part of the general theory, proofs for Theorems A and G are obtained in Section 5. We then prove Theorem C

⁶Another less central axiom, the Embedding Axiom, is satisfied by L^p as a bifunctor on Fréchet spaces and L_{rc}^{∞} as a bifunctor on integral complete locally convex spaces (but possibly not by R or by L^p as a bifunctor on other spaces). Beyond Fréchet spaces, the axiom is not used for major results.

and further results on Banach-Lie groups and local Banach-Lie groups (Section 6). Next, we study measurable regularity properties of projective limits of Lie groups (Section 7) and of ascending unions of Lie groups (Section 8), including proofs for Theorems D, E and F. In Section 9, we prove the L^1 -regularity of $\text{Diff}_K(\mathbb{R}^n)$, $\text{Diff}(\mathbb{S}_1)$, and $\text{Diff}_c(\mathbb{R}^n)$, before proving Theorem B (in full generality) in Section 11. The proof uses some basic uniqueness results for solutions to initial value problems with measurable right hand sides, provided in Section 10. We then establish Theorems H and I (Section 12). The proofs for the preparatory Section 1 have been relegated to an appendix (Appendix A), as well as proofs of auxiliary results on Lebesgue spaces of projective limits needed in Section 7 (see Appendix B) and some calculations concerning diffeomorphism groups (Appendix C).

Acknowledgement. The author thanks K.-H. Neeb, who suggested the consideration of measurable regularity properties and mentioned potential relations to control theory as well as the Trotter product and commutator formulas.

1 Preliminaries and notation

In this section, we fix our notation and terminology concerning topology, infinite-dimensional calculus and Lebesgue spaces of vector-valued measurable mappings. Several basic facts will be stated for later use. Many of these are easy to take on faith, whence we relegate proofs to the appendix (Appendix A).

We write $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If $f: X \rightarrow Y$ is a function, we write $\text{graph}(f) := \{(x, f(x)): x \in X\}$ for its graph. All vector spaces encountered in the article are real vector spaces, unless we explicitly say they are complex vector spaces. We use ‘locally convex space’ as a shorthand for ‘locally convex topological vector space.’ All topological spaces and locally convex spaces occurring in the article are assumed Hausdorff, except for the \mathcal{L}^p -spaces and \mathcal{L}_{rc}^∞ -spaces presently encountered, which are merely a preliminary for the definition of the Hausdorff L^p -spaces (and L_{rc}^∞ -spaces) we are really interested in. If (X, d) is a metric space, $x \in X$ and $r > 0$, we write

$$B_r^d(x) := \{y \in X: d(x, y) < r\} \quad \text{and} \quad \overline{B}_r^d(x) := \{y \in X: d(x, y) \leq r\}$$

for the open ball and closed ball, respectively. If q is a seminorm on a vector space E , we write

$$B_r^q(x) := \{y \in E : q(y - x) < r\} \quad \text{and} \quad \overline{B}_r^q(x) := \{y \in E : q(y - x) \leq r\}$$

for $x \in E$, $r > 0$. If $E = \mathbb{R}^n$, we let $\|\cdot\|_\infty$ be the maximum norm on \mathbb{R}^n and abbreviate $B_r(x) := B_r^{\|\cdot\|_\infty}(x)$ as well as $\overline{B}_r(x) := \overline{B}_r^{\|\cdot\|_\infty}(x)$. If $\lambda : E \rightarrow F$ is a continuous linear map between normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, we write $\|\lambda\|_{op}$ for its operator norm. Some basic concepts and facts from topology will be useful.

1.1 We recall the Wallace Lemma [43, Chapter 5, Theorem 12]:

Let X_1 and X_2 be topological spaces, $K_1 \subseteq X_1$ and $K_2 \subseteq X_2$ be compact subsets and $U \subseteq X_1 \times X_2$ be an open set such that $K_1 \times K_2 \subseteq U$. Then there exist open subsets $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ such that $K_1 \times K_2 \subseteq U_1 \times U_2 \subseteq U$.

1.2 Let (J, \leq) be a directed set, $((X_j)_{j \in J}, (\phi_{i,j})_{i \leq j})$ be a projective system⁷ of topological spaces X_j and continuous mappings $\phi_{i,j} : X_j \rightarrow X_i$ for $i, j \in J$ such that $i \leq j$. Let $(X, (\phi_j)_{j \in J})$ be a projective limit of the above system in the category of topological spaces and continuous mappings.⁸ The $\phi_{i,j}$ will be called *bonding maps*, and the ϕ_j will be called *limit maps*. Then the following holds:

- (a) *For each $x \in X$, the sets $\phi_j^{-1}(U)$ form a basis of open neighbourhoods of x , for $j \in J$ and U ranging through the set of open neighbourhoods of $\phi_j(x)$ in X_j .*
- (b) *A subset $D \subseteq X$ is dense in X if and only if $\phi_j(D)$ is dense in $\phi_j(X)$ for each $j \in J$.*

1.3 By a *topological embedding*, we mean a map $f : X \rightarrow Y$ between topological spaces such that the co-restriction $f|^{f(X)} : X \rightarrow f(X)$ is a homeomorphism with respect to the topology induced by Y on the image $f(X) = \text{im}(f)$.

1.4 A topology on a real vector space E is called a *vector topology* if it turns E into a topological vector space.

⁷Thus $\phi_{j,j} = \text{id}_{X_j}$ for all $j \in J$ and $\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k}$ for all $i, j, k \in J$ such that $i \leq j \leq k$.

⁸That is, the map $(\phi_j)_{j \in J} : X \rightarrow \{(x_j)_{j \in J} \in \prod_{j \in J} X_j : (\forall i, j \in J) i \leq j \Rightarrow x_i = \phi_{i,j}(x_j)\}$ is a homeomorphism.

1.5 Throughout, we are working in the setting of abstract set theory. Thus, if (X_1, Σ_1) and (X_2, Σ_2) are measurable spaces (viz. X_i is a set and Σ_i a σ -algebra on X_i), we call a map $f: X_1 \rightarrow X_2$ measurable if $f^{-1}(A) \in \Sigma_1$ for all $A \in \Sigma_2$ (see, e.g., [3, §7]). We also say that $f: (X_1, \Sigma_1) \rightarrow (X_2, \Sigma_2)$ is measurable in the situation. If \mathcal{E} is a set of subset of a set X , we write $\sigma(\mathcal{E})$ for the σ -algebra on X generated by \mathcal{E} . As usual, if X is a topological space, we write $\mathcal{B}(X) := \sigma(\mathcal{O})$ for the σ -algebra of Borel sets of X , generated by the set \mathcal{O} of all open subsets of X . If (X, Σ) is a measurable space and Y a subset of X , then the trace

$$\Sigma|_Y := \{A \cap Y : A \in \Sigma\}$$

is a σ -algebra on Y . If $Y \in \Sigma$, then

$$\Sigma|_Y = \{A \in \Sigma : A \subseteq Y\}$$

(see [3, §1]). If (X_i, Σ_i) are measurable spaces, as usual we let $\Sigma_1 \otimes \Sigma_2$ be the product σ -algebra, i.e., the σ -algebra on $X_1 \times X_2$ generated by $\{A_1 \times A_2 : A_1 \in \Sigma_1, A_2 \in \Sigma_2\}$.

Some simple basic facts will be used:

- 1.6** (a) If (X, Σ) is a measurable space and $Y \subseteq X$, then the inclusion map $j: (Y, \Sigma|_Y) \rightarrow (X, \Sigma)$ is measurable (as $j^{-1}(A) = A \cap Y \in \Sigma|_Y$ for each $A \in \Sigma$) and hence $f|_Y = f \circ j: (Y, \Sigma|_Y) \rightarrow (X_2, \Sigma_2)$ is measurable for each measurable map $f: (X, \Sigma) \rightarrow (X_2, \Sigma_2)$ to a measurable space.
- (b) Let (X_i, Σ_i) be measurable spaces for $i \in \{1, 2\}$ and $f: X_1 \rightarrow X_2$ be a map such that $f(X_1) \subseteq Y$ for a subset $Y \subseteq X_2$. Then $f: (X_1, \Sigma_1) \rightarrow (X_2, \Sigma_2)$ is measurable if and only if the co-restriction $f|_Y: (X_1, \Sigma_1) \rightarrow (Y, \Sigma_2|_Y)$ is measurable.⁹
- (c) Let X be a set and \mathcal{E} be a set of subsets of X . Then $\sigma(\mathcal{E})|_Y = \sigma(\{A \cap Y : A \in \mathcal{E}\})$ for each subset $Y \subseteq X$. In particular:
- (d) If X is a topological space and we endow a subset $Y \subseteq X$ with the induced topology, then $\mathcal{B}(Y) = \mathcal{B}(X)|_Y$. Hence, if $Y \in \mathcal{B}(X)$, then $\mathcal{B}(Y) = \{A \in \mathcal{B}(X) : A \subseteq Y\}$.

⁹If $f|_Y$ is measurable, then also $f = j \circ f|_Y$ with the measurable inclusion map $j: Y \rightarrow X_2$. If f is measurable, then each $B \in \Sigma_2|_Y$ is of the form $B = Y \cap A$ with some $A \in \Sigma_2$ and hence $(f|_Y)^{-1}(B) = f^{-1}(B \cap f(X_1)) = f^{-1}(A \cap f(X_1)) = f^{-1}(A) \in \Sigma_1$, showing that $f|_Y$ is measurable.

- (e) Let (X, Σ) be a measurable space, Y be a set and \mathcal{E} be a set of subsets of Y . Then a map $f: (X, \Sigma) \rightarrow (Y, \sigma(\mathcal{E}))$ is measurable if and only if $f^{-1}(A) \in \Sigma$ for each $A \in \mathcal{E}$ [3, Satz 7.2]. In particular, a map

$$f = (f_1, f_2): (X, \Sigma) \rightarrow (X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$$

is measurable if and only if both $f_1: (X, \Sigma) \rightarrow (X_1, \Sigma_1)$ and $f_2: (X, \Sigma) \rightarrow (X_2, \Sigma_2)$ are measurable.

- (f) If X_1, X_2 are topological spaces and X_2 has a countable basis for its topology, then $\mathcal{B}(X_1 \times X_2) = \mathcal{B}(X_1) \otimes \mathcal{B}(X_2)$ (see, e.g., [24, Lemma 2.7]).

For a classical discussion of vector-valued integrals in Banach spaces, see [41, Chapter III]; see also [62] for further relevant aspects. It is essential for our purposes to go beyond the classical frame of Banach spaces and consider \mathcal{L}^p -functions (and related functions) also with values in Fréchet spaces (and, later, in even more general locally convex spaces).

1.7 Let (X, Σ, μ) be a measure space, E be a Fréchet space, $p \in [1, \infty[$ and $P(E)$ be the set of all continuous seminorms $q: E \rightarrow [0, \infty[$. We let $\mathcal{L}^p(X, \mu, E)$ be the set of all measurable functions $\gamma: (X, \Sigma) \rightarrow (E, \mathcal{B}(E))$ such that

$$(a) \quad \|\gamma\|_{\mathcal{L}^p, q} := \sqrt[p]{\int_X q(\gamma(x))^p d\mu(x)} < \infty \text{ for each } q \in P(E); \text{ and}$$

- (b) The image $\gamma(X)$ is separable (i.e., it has a dense countable subset).

1.8 For E a Fréchet space, let $\mathcal{L}^\infty(X, \mu, E)$ be the set of all measurable mappings $\gamma: (X, \Sigma) \rightarrow (E, \mathcal{B}(E))$ with separable, bounded image. For $\gamma \in \mathcal{L}^\infty(X, \mu, E)$ and q a continuous seminorm on E , we write

$$\|\gamma\|_{\mathcal{L}^\infty, q} := \|q \circ \gamma\|_{\mathcal{L}^\infty} = \text{ess sup}_\mu (q \circ \gamma). \quad (3)$$

Let $\mathcal{L}_{rc}^\infty(X, \mu, E)$ be the set of all measurable maps $\gamma: (X, \Sigma) \rightarrow (E, \mathcal{B}(E))$ with relatively compact image. Linear combinations of measurable E -valued maps with separable image being measurable (cf. Lemma A.1), $\mathcal{L}^q(X, \mu, E)$, $\mathcal{L}^\infty(X, \mu, E)$ and $\mathcal{L}_{rc}^\infty(X, \mu, E)$ are vector subspaces of E^X . As is clear,

$$\mathcal{L}_{rc}^\infty(X, \mu, E) \subseteq \mathcal{L}^\infty(X, \mu, E), \quad (4)$$

with equality if and only if all bounded subsets of E are relatively compact, i.e., E is semi-Montel. This holds for example if E is Schwartz, nuclear or finite-dimensional.¹⁰ If μ is a finite measure, then

$$\mathcal{L}^\infty(X, \mu, E) \subseteq \mathcal{L}^p(X, \mu, E) \subseteq \mathcal{L}^1(X, \mu, E) \text{ for all } p \in [1, \infty] \quad (5)$$

For $p \in [1, \infty]$, we endow $\mathcal{L}^p(X, \mu, E)$ with the (not necessarily Hausdorff) locally convex vector topology defined by the seminorms $\|\cdot\|_{\mathcal{L}^p, q}$, for $q \in P(E)$. We give $\mathcal{L}_{rc}^\infty(X, \mu, E)$ the topology induced by $\mathcal{L}^\infty(X, \mu, E)$. Then the inclusion maps in (4) and (5) are continuous.

1.9 If E is an arbitrary locally convex space, then $\mathcal{L}_{rc}^\infty(X, \mu, E)$ is defined as the set of all measurable functions $\gamma: X \rightarrow E$ such that the closure $\overline{\text{im}(\gamma)}$ of the image of γ in E is compact and metrizable [24]. Also in this generality, $\mathcal{L}_{rc}^\infty(X, \mu, E)$ is a vector subspace of E^X (see [24]). We define seminorms $\|\cdot\|_{\mathcal{L}^\infty, q}$ on $\mathcal{L}_{rc}^\infty(X, \mu, E)$ as in (3) and use these to endow $\mathcal{L}_{rc}^\infty(X, \mu, E)$ with a vector topology.

The following two lemmas are useful tools for dealing with the mappings $\gamma \in \mathcal{L}_{rc}^\infty(X, \mu, E)$. See, e.g., [24, Lemma 2.1] for the first fact (which can be deduced from [17, Theorem 4.2.13]):

Lemma 1.10 *If K is a metrizable compact topological space and $f: K \rightarrow Y$ a continuous map to a Hausdorff topological space Y , then also the image $f(K)$ is compact and metrizable.* \square

Lemma 1.11 *Let E be a locally convex space and $K \subseteq E$ a subset which is compact and metrizable in the induced topology. Let $E_K := \text{span}(K) \subseteq E$ be the vector subspace spanned by K and \mathcal{O}_K be the induced topology on E_K . Then E_K can be given a separable metrizable locally convex vector topology \mathcal{O}' such that $\mathcal{O}' \subseteq \mathcal{O}_K$.*

As before, (X, Σ, μ) is a measure space.

1.12 If Y is a topological space and $\gamma: X \rightarrow Y$ a measurable map, we write $[\gamma]$ for the set of all measurable mappings $\gamma_1: X \rightarrow Y$ such that $\gamma(x) = \gamma_1(x)$ for μ -almost all $x \in X$.

¹⁰As every Fréchet space is barrelled, it is semi-Montel iff it is Montel.

1.13 If E is a Fréchet space and $p \in [1, \infty]$, we set

$$L^p(X, \mu, E) := \{[\gamma] : \gamma \in \mathcal{L}^p(X, \mu, E)\}.$$

Let $N_p \subseteq \mathcal{L}^p(X, \mu, E)$ be the vector subspaces of all γ in $\mathcal{L}^p(X, \mu, E)$ such that $\gamma(x) = 0$ for μ -almost all $x \in X$. Then the map

$$\mathcal{L}^p(X, \mu, E)/N_p \rightarrow L^p(X, \mu, E), \quad \gamma + N_p \rightarrow [\gamma] \quad (6)$$

is a bijection, which we use to identify $L^p(X, \mu, E)$ with the quotient vector space $\mathcal{L}^p(X, \mu, E)/N_p$. Similarly, for E a locally convex space we let $N_{rc} \subseteq \mathcal{L}_{rc}^\infty(X, \mu, E)$ be the vector space of all $\gamma \in \mathcal{L}_{rc}^\infty(X, \mu, E)$ such that $\gamma(x) = 0$ for μ -almost all $x \in X$. We use the map

$$\mathcal{L}_{rc}^\infty(X, \mu, E)/N_{rc} \rightarrow L_{rc}^\infty(X, \mu, E), \quad \gamma + N_{rc} \rightarrow [\gamma]$$

to identify the quotient vector space $\mathcal{L}_{rc}^\infty(X, \mu, E)/N_{rc}$ with $L_{rc}^\infty(X, \mu, E) := \{[\gamma] : \gamma \in \mathcal{L}_{rc}^\infty(X, \mu, E)\}$.

1.14 If E is a Fréchet space, we obtain a seminorm $\|\cdot\|_{L^p, q}$ on $L^p(X, \mu, E)$ via $\|[\gamma]\|_{L^p, q} := \|\gamma\|_{\mathcal{L}^p, q}$, for each continuous seminorm q on E . We give $L^p(X, \mu, E)$ the locally convex vector topology defined by the seminorms $\|\cdot\|_{L^p, q}$, for $q \in P(E)$. Likewise, for E a locally convex space, we make $L_{rc}^\infty(X, \mu, E)$ a locally convex space using the seminorms $\|\cdot\|_{L^\infty, q}$ defined via $\|[\gamma]\|_{L^\infty, q} := \|\gamma\|_{\mathcal{L}^\infty, q}$. The latter topologies coincide with the quotient topologies on the quotient spaces $L^p(X, \mu, E) = \mathcal{L}^p(X, \mu, E)/N_p$ and $L_{rc}^\infty(X, \mu, E) = \mathcal{L}_{rc}^\infty(X, \mu, E)/N_{rc}$, respectively.

1.15 If E is a Fréchet space, then

$$L_{rc}^\infty(X, \mu, E) \subseteq L^\infty(X, \mu, E)$$

and the above topology on $L_{rc}^\infty(X, \mu, E)$ coincides with the induced topology. If μ is a finite measure and E a Fréchet space, then

$$L^\infty(X, \mu, E) \subseteq L^p(X, \mu, E)$$

for all $p \in [1, \infty[$, and the inclusion map is linear and continuous as $\|[\gamma]\|_{L^p, q} \leq \sqrt[p]{\mu(X)} \|[\gamma]\|_{L^\infty, q}$. We also have

$$L^{p_1}(X, \mu, E) \subseteq L^{p_2}(X, \mu, E)$$

for all $p_1 \geq p_2$ in $[1, \infty[$, and the inclusion map is continuous linear.¹¹

1.16 We recall the concept of weak integral. Let (X, Σ, μ) be a measure space, E be a locally convex space, E' be the space of all continuous linear functionals $\lambda: E \rightarrow \mathbb{R}$ and $\gamma: X \rightarrow E$ be a function such that $\lambda \circ \gamma \in \mathcal{L}^1(X, \mu, \mathbb{R})$ for each $\lambda \in E'$. An element $w \in E$ is called the *weak integral of γ with respect to μ* (and denoted $\int_X \gamma(x) d\mu(x) := w$) if

$$(\forall \lambda \in E') \quad \lambda(w) = \int_X \lambda(\gamma(x)) d\mu(x).$$

1.17 The following is clear, as $\lambda \circ \alpha \in E'$ for each $\lambda \in F'$:

If $\gamma: X \rightarrow E$ has a weak integral $\int_X \gamma(x) d\mu(x)$ in the situation of 1.16, and $\alpha: E \rightarrow F$ is a continuous linear map to a locally convex space F , then $\alpha \circ \gamma: X \rightarrow F$ has a weak integral in F ; in fact,

$$\int_X \alpha(\gamma(x)) d\mu(x) = \alpha \left(\int_X \gamma(x) d\mu(x) \right),$$

as the right-hand side satisfies the property which defines the weak integral on the left.

1.18 We shall use continuity of parameter-dependent integrals (see, e.g., [6, Proposition 3.5], or [38]):

Let E be a locally convex space, X a topological space, $a, b \in \mathbb{R}$ such that $a < b$ and $f: X \times [a, b] \rightarrow E$ be a continuous function such that the weak integral

$$g(x) := \int_a^b f(x, t) dt$$

exists in E for each $x \in X$. Then $g: X \rightarrow E$ is continuous.

The next lemma compiles essential basic properties of the spaces from 1.7.

¹¹In fact, $q(\gamma(x))^{p_2} \leq \max\{1, r^{p_1} q(\gamma(x))^{p_1}\}$ and thus $\|\gamma\|_{L^{p_2, q}} \leq (\|1 + (q \circ \gamma)\|_{L^{p_1}})^{p_1/p_2} \leq (\|1\|_{L^{p_1}} + \|q \circ \gamma\|_{L^{p_1}})^{p_1/p_2} = (\sqrt[p_1]{\mu(X)} + \|q \circ \gamma\|_{L^{p_1}})^{p_1/p_2}$. Set $C := (\sqrt[p_1]{\mu(X)} + 1)^{p_1/p_2}$. By the preceding, $\|\gamma\|_{L^{p_2, q}} \leq C$ for all $[\gamma] \in L^{p_1}(X, \mu, E)$ such that $\|\gamma\|_{L^{p_1, q}} \leq 1$. Hence $\|\gamma\|_{L^{p_2, q}} \leq C \|\gamma\|_{L^{p_1, q}}$ for all $[\gamma] \in L^{p_1}(X, \mu, E)$.

Lemma 1.19 *Let (X, Σ, μ) be a measure space and E be a Fréchet space (resp., a Banach space). Let $p \in [1, \infty]$. Then $L^p(X, \mu, E)$ and $L_{rc}^\infty(X, \mu, E)$ are Fréchet spaces (resp., Banach spaces). Moreover, each $\gamma \in \mathcal{L}^1(X, \mu, E)$ admits a weak integral $\int_X \gamma(x) d\mu(x)$, and the map*

$$I: L^1(X, \mu, E) \rightarrow E, \quad [\gamma] \mapsto \int_X \gamma(x) d\mu(x)$$

is well-defined and continuous linear, with $q(I(\gamma)) \leq \|[\gamma]\|_{L^1, q}$ for each continuous seminorm q on E .

1.20 We say that a locally convex space E is *integral complete* if each continuous curve $\gamma: [0, 1] \rightarrow E$ has a weak integral $\int_a^b \gamma(t) dt \in E$. It is known that E is integral complete if and only if E has the *metric convex compactness property* (metric CCP) in the sense that the closed convex hull

$$\overline{\text{conv}(K)} \subseteq E$$

is compact for each compact metrizable subset $K \subseteq E$ (see [71]; cf. [70]).

Lemma 1.21 *If E is a locally convex space and $K \subseteq E$ a metrizable compact subset such that $\overline{\text{conv}(K)} \subseteq E$ is compact, then $\overline{\text{conv}(K)}$ is metrizable.*

1.22 If $\mu(X) < \infty$, we can define $\|[\gamma]\|_{L^1, q} := \|\gamma\|_{\mathcal{L}^1, q} \in [0, \infty[$ as in 1.7 (a) also for E an arbitrary locally convex space and $\gamma \in \mathcal{L}_{rc}^\infty(X, \mu, E)$.

Lemma 1.23 *Let (X, Σ, μ) be a measure space and E be an integral complete locally convex space. If $\mu(X) < \infty$, then each $\gamma \in \mathcal{L}_{rc}^\infty(X, \mu, E)$ admits a weak integral $\int_X \gamma(x) d\mu(x)$, and the map*

$$I: L_{rc}^\infty(X, \mu, E) \rightarrow E, \quad [\gamma] \mapsto \int_X \gamma(x) d\mu(x)$$

is well-defined and continuous linear, with $q(I(\gamma)) \leq \|[\gamma]\|_{L^1, q} \leq \mu(X) \|[\gamma]\|_{L^\infty, q}$ for each continuous seminorm q on E .

1.24 If $J \subseteq \mathbb{R}$ is an interval and μ the restriction of the 1-dimensional Lebesgue-Borel measure λ_1 to $\mathcal{B}(J)$, we omit mention of μ and simply write $\mathcal{L}^p(J, E)$ instead of $\mathcal{L}^p(J, \mu, E)$, etc. If, moreover, $J = [a, b]$ with reals $a \leq b$, we write $\int_a^b \gamma(t) dt$ in place of $\int_{[a, b]} \gamma(t) d\mu(t)$, for $\gamma \in \mathcal{L}^1([a, b], E)$. If $a > b$, we define $\int_a^b \gamma(t) dt := - \int_b^a \gamma(t) dt$ for $\gamma \in \mathcal{L}^1([b, a], E)$. Likewise for $\mathcal{L}_{rc}^\infty(J, E)$ if E is an integral complete locally convex space.

1.25 Because the map $C(J, E) \rightarrow L^\infty(J, E)$, $\gamma \mapsto [\gamma]$ is injective, we can identify $\gamma \in C(J, E)$ with its equivalence class $[\gamma]$ in $L_{rc}^\infty(J, E)$, for E an arbitrary locally convex space and $J \subseteq \mathbb{R}$ an interval. Then

$$\|\gamma\|_{\mathcal{L}^\infty, q} = \sup_{t \in J} q(\gamma(t)),$$

for each $\gamma \in C(J, E)$ and each continuous seminorm q on E . Likewise, we can identify $\gamma \in C(J, E) \cap \mathcal{L}^p(J, E)$ with its coset $[\gamma]$ in $L^p(J, E)$, for each Fréchet space E , $\gamma \in C(J, E)$ and $p \in [1, \infty]$.

Remark 1.26 In this section, we clearly distinguish between a measurable function γ and its equivalence class $[\gamma]$ under equality μ -almost everywhere. Later on, when the meaning is clear from the context, we shall sometimes ignore the distinction.

1.27 Let $J \subseteq \mathbb{R}$ be non-degenerate interval, E be a locally convex space, $\eta: J \rightarrow E$ be a mapping and $t \in J$. As usual, we say that η is *differentiable at t* if the limit $\eta'(t) = \lim_{s \rightarrow t} \frac{\eta(s) - \eta(t)}{s - t}$ exists in E . Then η is continuous at t in particular.¹²

The following version of the Fundamental Theorem of Calculus is essential. Compare [62, 25.16] if E is a Banach space.

Lemma 1.28 *Let $J \subseteq \mathbb{R}$ be an interval, E be a Fréchet space and $t_0 \in J$. If $\gamma \in \mathcal{L}^1(J, E)$, then the weak integrals needed to define*

$$\eta: J \rightarrow E, \quad t \mapsto \int_{t_0}^t \gamma(s) ds$$

exist, and η is a continuous function which is differentiable λ_1 -almost everywhere, with $\eta' = [\gamma]$.

For general locally convex spaces E , we still have the following.

Lemma 1.29 *Let $J \subseteq \mathbb{R}$ be an interval, E be an integral complete locally convex space and $t_0 \in J$. If $\gamma \in \mathcal{L}_{rc}^\infty(J, E)$, then the weak integrals needed to define*

$$\eta: J \rightarrow E, \quad t \mapsto \int_{t_0}^t \gamma(s) ds$$

exist, and η is a continuous function. If also $\gamma_1 \in \mathcal{L}_{rc}^\infty(J, E)$ is a map such that $\eta(t) = \int_{t_0}^t \gamma_1(s) ds$ for all $t \in J$, then $[\gamma] = [\gamma_1]$.

¹²Indeed, $\eta(s) = \eta(t) + (s - t) \frac{\eta(s) - \eta(t)}{s - t} \rightarrow \eta(t) + 0\eta'(t) = \eta(t)$ as $s \rightarrow t$.

1.30 In the situation of Lemmas 1.28 and 1.29, respectively, we shall write $\eta' := [\gamma]$. Since $[\gamma]$ is uniquely determined by η in Lemma 1.29 (and $\gamma(t) = \eta'(t)$ λ_1 -almost everywhere in Lemma 1.28), we see that $\eta' \in L_{rc}^\infty(J, E)$ (resp., $\eta' \in L^1(J, E)$) is well-defined.

If E is a Fréchet space, then a function $\gamma: X \rightarrow E$ is in $\mathcal{L}_{rc}^\infty(X, \mu, E)$ if and only if there exists a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of measurable functions $\gamma_n: X \rightarrow E$ with finite image such that $\gamma_n \rightarrow \gamma$ uniformly (see [24, Corollary 3.19]).¹³ Passing to more restrictive functions γ_n (the step functions), we arrive at the notion of regulated functions.

1.31 Let E be a locally convex space and $a, b \in \mathbb{R}$ with $a < b$. Let $\mathcal{T}([a, b], E)$ be the set of all functions $\gamma: [a, b] \rightarrow E$ for which there exists a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

of $[a, b]$ such that $\gamma|_{[t_{j-1}, t_j]}$ is constant for all $j \in \{1, \dots, n\}$. We let

$$\mathcal{R}([a, b], E) \subseteq \mathcal{L}_{rc}^\infty([a, b], E)$$

be the space of functions $\gamma: [a, b] \rightarrow E$ which are the uniform limit of a sequence $(\gamma_n)_{n \in \mathbb{N}}$ in $\mathcal{T}([a, b], E)$. Such functions are called *regulated functions* from $[a, b]$ to E . Then $\mathcal{R}([a, b], E)$ is a vector subspace of $\mathcal{L}_{rc}^\infty([a, b], E)$ and $R([a, b], E) := \{[\gamma]: \gamma \in \mathcal{R}([a, b], E)\}$ is a vector subspace of $L_{rc}^\infty([a, b], E)$. We endow both vector subspaces with the induced topology.

We mention:

Lemma 1.32 *If E is a locally convex space, $a < b$ in \mathbb{R} and $\gamma \in \mathcal{R}([a, b], E)$, then the sequence $(\gamma_n)_{n \in \mathbb{N}}$ in $\mathcal{T}([a, b], E)$ such that $\gamma_n \rightarrow \gamma$ uniformly can be chosen such that $\gamma_n([a, b]) \subseteq \gamma([a, b])$ for all $n \in \mathbb{N}$.*

1.33 If E is a Fréchet space, then $R([a, b], E)$ is the closure of (equivalence classes of) $\mathcal{T}([a, b], E)$ in $L_{rc}^\infty([a, b], E)$. Thus $R([a, b], E)$ is a Fréchet space.

1.34 Let (X, Σ, μ) be a measure space. If $\lambda: E \rightarrow F$ is a continuous linear map between Fréchet spaces, then

$$L^p(X, \mu, \lambda): L^p(X, \mu, E) \rightarrow L^p(X, \mu, F), \quad [\gamma] \mapsto [\lambda \circ \gamma]$$

¹³If γ_n exist, then even γ_n with $\text{im } \gamma_n \subseteq \text{im } \gamma$ exist, as we may choose $v_1, \dots, v_{m(n)} \in \text{im}(\gamma)$ in the proof of [24, Proposition 3.18]. We shall not use this fact.

(and the map $\mathcal{L}^p(X, \mu, \lambda): \mathcal{L}^p(X, \mu, E) \rightarrow \mathcal{L}^p(X, \mu, F)$, $\gamma \mapsto \lambda \circ \gamma$) is continuous linear, for each $p \in [1, \infty]$. In fact, if q is a continuous seminorm on F , then $q \circ \lambda$ is a continuous seminorm on E and $\|\lambda \circ \gamma\|_{\mathcal{L}^p, q} = \|\gamma\|_{\mathcal{L}^p, q \circ \lambda} \leq \|\gamma\|_{\mathcal{L}^p, q \circ \lambda}$ for all $\gamma \in \mathcal{L}^p(X, \mu, E)$, entailing that the linear maps $\mathcal{L}^p(X, \mu, \lambda)$ and $L^p(X, \mu, \lambda)$ are continuous. Similarly, the linear maps

$$L_{rc}^\infty(X, \mu, \lambda): L_{rc}^\infty(X, \mu, E) \rightarrow L_{rc}^\infty(X, \mu, F), \quad [\gamma] \mapsto [\lambda \circ \gamma]$$

and $\mathcal{L}_{rc}^\infty(X, \mu, \lambda): \mathcal{L}_{rc}^\infty(X, \mu, E) \rightarrow \mathcal{L}_{rc}^\infty(X, \mu, F)$, $\gamma \mapsto \lambda \circ \gamma$ are continuous linear for each linear map $\lambda: E \rightarrow F$ between locally convex spaces, and are the maps

$$R([a, b], \lambda): R([a, b], E) \rightarrow R([a, b], F), \quad [\gamma] \mapsto [\lambda \circ \gamma]$$

and $\mathcal{R}([a, b], E) \rightarrow \mathcal{R}([a, b], F)$, $\gamma \mapsto \lambda \circ \gamma$.

1.35 As a consequence of 1.34, we have that

$$L^p(X, \mu, E_1 \times E_2) \cong L^p(X, \mu, E_1) \times L^p(X, \mu, E_2)$$

for all $p \in [1, \infty]$ and Fréchet spaces E_1 and E_2 . Similarly,

$$L_{rc}^\infty(X, \mu, E_1 \times E_2) \cong L_{rc}^\infty(X, \mu, E_1) \times L_{rc}^\infty(X, \mu, E_2)$$

for all locally convex spaces E_1 and E_2 and $R([a, b], E_1 \times E_2) \cong R([a, b], E_1) \times R([a, b], E_2)$.

The following two lemmas will help us to deal with locally convex direct sums and locally convex direct limits.

Lemma 1.36 *Let $(E_n)_{n \in \mathbb{N}}$ be a sequence of locally convex spaces. Fix $p \in [0, \infty]$. Then*

$$\bigoplus_{n \in \mathbb{N}} E_n \rightarrow [0, \infty[, \quad (x_n)_{n \in \mathbb{N}} \mapsto \|(q_n(x_n))_{n \in \mathbb{N}}\|_{\ell^p} \quad (7)$$

is a continuous seminorm on the locally convex direct sum, for each sequence $(q_n)_{n \in \mathbb{N}}$ of continuous seminorms $q_n: E_n \rightarrow [0, \infty[$. The locally convex direct sum topology on $\bigoplus_{n \in \mathbb{N}} E_n$ is defined by the set of seminorms of the form (7).

More explicitly, the seminorms are given by

$$\left(\sum_{n=1}^{\infty} (q_n(x_n))^p \right)^{1/p}$$

if $p < \infty$. For $p = \infty$, they are given by

$$\sup\{q_n(x_n) : n \in \mathbb{N}\}.$$

Lemma 1.37 *Let $E_1 \subseteq E_2 \subseteq \dots$ be an ascending sequence of locally convex spaces such that all inclusion maps $E_n \rightarrow E_{n+1}$ are continuous linear. Endow $E := \bigcup_{n \in \mathbb{N}} E_n$ with the (not necessarily Hausdorff) locally convex direct limit topology. Fix $p \in [1, \infty]$. Then the sets*

$$\left\{ \sum_{n=1}^{\infty} v_n : (v_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} E_n \text{ such that } \|(q_n(v_n))_{n \in \mathbb{N}}\|_{\ell^p} < 1 \right\}$$

form a basis of 0-neighbourhoods for E , if we let $(q_n)_{n \in \mathbb{N}}$ run through the set of all sequences of continuous seminorms q_n on E_n .

Also beyond metric spaces, let us say that a topological space X is *separable* if it contains a countable dense subset.

Definition 1.38 Let us say that a locally convex space E has the *Fréchet exhaustion property* (FEP) if every separable closed vector subspace $S \subseteq E$ can be written as a union $S = \bigcup_{n \in \mathbb{N}} F_n$ of vector subspaces $F_1 \subseteq F_2 \subseteq \dots$ of E which are Fréchet spaces in the induced topology. In this case, we also say that E is a (FEP)-space.

For example, every Fréchet space has the Fréchet exhaustion property (as we can take $F_n := S$ for all $n \in \mathbb{N}$).

By the next lemma, the Fréchet spaces F_n in Definition 1.38 are separable.

Lemma 1.39 *Let E be a locally convex space and $F \subseteq E$ be a vector subspace, endowed with the induced topology. If E is separable and F is metrizable, then also F is separable.*

1.40 If a locally convex space E has the Fréchet exhaustion property and (X, Σ, μ) is a measure space, then the measurable functions $\gamma: X \rightarrow E$ with separable image and

$$\|\gamma\|_{\mathcal{L}^{p,q}} := \|q \circ \gamma\|_{\mathcal{L}^p} < \infty$$

for all $q \in P(E)$ form a vector space $\mathcal{L}^p(X, \mu, E)$, giving rise to a Hausdorff locally convex space $L^p(X, \mu, E)$ of equivalence classes (as we prove in Appendix A).¹⁴ As in the Fréchet case, we write $\|[\gamma]\|_{L^p,q} := \|\gamma\|_{\mathcal{L}^{p,q}}$.

Lemma 1.41 (a) *Every locally convex direct sum $E := \bigoplus_{j \in J} E_j$ of sequentially complete (FEP)-spaces¹⁵ is sequentially complete and has the (FEP). In this case,*

$$L^p(X, \mu, E) = \bigoplus_{j \in J} L^p(X, \mu, E_j)$$

as a vector space, for each $p \in [1, \infty]$ and measure space (X, Σ, μ) . Moreover,

$$L^1(X, \mu, E) = \bigoplus_{j \in J} L^1(X, \mu, E_j)$$

as a locally convex space.

(b) *If a locally convex space E has the (FEP), then every closed vector subspace $F \subseteq E$ has the (FEP).*

(c) *Every strict (LF)-space $E = \varinjlim E_n$ has the (FEP). In this case,*

$$L^p(X, \mu, E) = \bigcup_{n \in \mathbb{N}} L^p(X, \mu, E_n)$$

as a vector space, for each $p \in [1, \infty]$ and measure space (X, Σ, μ) .

(d) *Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\Gamma_c^{C^k}(V)$ be the space of compactly supported C^k -sections in a vector bundle V over a paracompact finite-dimensional smooth manifold M , whose typical fibre F is a Fréchet space.¹⁶ Then $\Gamma_c^{C^k}(V)$ has the (FEP), and*

$$L^p(X, \mu, \Gamma_c^{C^k}(V)) = \bigcup_K L^p(X, \mu, \Gamma_K^{C^k}(V))$$

¹⁴If E does not have the (FEP), then there is no reason why $\mathcal{L}^p(X, \mu, E)$ should be closed under sums (as sums might not be measurable); it might not be a vector space.

¹⁵This condition is satisfied, e.g., if each E_j is a Fréchet space.

¹⁶The topology on this space is recalled in Appendix A, see A.13.

for each $p \in [1, \infty]$ and measure space (X, Σ, μ) , where K ranges through the set of compact subsets of M and

$$\Gamma_K^{C^k}(V) := \{\sigma \in C_c^{C^k}(M) : (\forall x \in M \setminus K) \sigma(x) = 0\}.$$

In particular:

- (e) The space $\mathcal{X}_c(M) = \Gamma_c(TM)$ of compactly supported smooth vector fields on a paracompact finite-dimensional smooth manifold M has the (FEP) (which is the modelling space of the Lie group $\text{Diff}_c(M)$).

Remark 1.42 Lemma 1.41 (c) will be strengthened further in Proposition 8.8 (a certain analogue of Mujica's Theorem on spaces of continuous functions to locally convex direct limits).

Lemma 1.43 *Let E be a locally convex space. If E is sequentially complete and has the Fréchet exhaustion property, then the weak integral*

$$\int_X \gamma d\mu$$

exists in E for each measure space (X, Σ, μ) and each $\gamma \in \mathcal{L}^1(X, \Sigma, \mu)$.

Given a measurable space (X, Σ) and a locally convex space E , we write $\mathcal{F}(X, E)$ for the space of all measurable functions $\gamma: X \rightarrow E$ with finite image. The following result is needed in Section 8.

Lemma 1.44 *Let E be a locally convex space with the (FEP) and (X, Σ, μ) be a measure space. Then $\mathcal{F}(X, E) \cap \mathcal{L}^p(X, \mu, E)$ is dense in $\mathcal{L}^p(X, \mu, E)$, for each $p \in [1, \infty[$. We even have*

$$\gamma \in \overline{\{\eta \in \mathcal{F}(X, E) \cap \mathcal{L}^p(X, \mu, E) : \eta(X) \subseteq \gamma(X) \cup \{0\}\}} \quad (8)$$

for each $\gamma \in \mathcal{L}^p(X, \mu, E)$.

1.45 If E is a locally convex space and $q \in P(E)$ a continuous seminorm on E , we set

$$E_q := E/q^{-1}(\{0\})$$

and abbreviate $[x] := x + q^{-1}(\{0\})$ for $x \in E$. It is well-known that E_q is a normed space with the norm

$$\|\cdot\|_q: E_q \rightarrow [0, \infty[, \quad \|[x]\|_q := q(x).$$

We let \tilde{E}_q be a completion of E_q such that $E_q \subseteq \tilde{E}_q$, and write $\|\cdot\|_q$ also for the extension of the given norm on E_q to a norm on \tilde{E}_q . Thus \tilde{E}_q is a Banach space, and

$$\pi_q: E \rightarrow \tilde{E}_q, \quad x \mapsto [x]$$

is a continuous linear map with dense image such that $\|\cdot\|_q \circ \pi_q = q$.

Lemma 1.46 *Let $J \subseteq \mathbb{R}$ be an interval, E be a sequentially complete locally convex space which has the Fréchet exhaustion property, and $t_0 \in J$. If $\gamma \in \mathcal{L}^1(J, E)$, then the weak integrals needed to define*

$$\eta: J \rightarrow E, \quad t \mapsto \int_{t_0}^t \gamma(s) ds$$

exist, and η is a continuous function which uniquely determines $[\gamma]$.

Again, we can therefore write $\eta' := [\gamma]$.

Lemma 1.47 *Let E be a Fréchet space and $\gamma \in \mathcal{L}^1([a, b], E)$; or let E be an integral complete locally convex space and $\gamma \in \mathcal{L}_{rc}^\infty([a, b], E)$. Let $F \subseteq E$ be a closed vector subspace and $\eta: [a, b] \rightarrow E$ be the map given by*

$$\eta(t) := \int_a^t \gamma(s) ds \quad \text{for } t \in [a, b].$$

Then $\eta([a, b]) \subseteq F$ if and only if $[\gamma] = [\gamma_1]$ for some $\gamma_1 \in \mathcal{L}^1([a, b], F)$ (resp., $\gamma_1 \in \mathcal{L}_{rc}^\infty([a, b], F)$). Likewise, we find $\gamma_1 \in \mathcal{L}^1([a, b], F)$ with $[\gamma] = [\gamma_1]$ if E is a strict (LF)-space, $F \subseteq E$ a vector subspace which is a Fréchet space in the induced topology, and $\gamma \in \mathcal{L}^1([a, b], E)$ with $\int_a^t \gamma(s) ds \in F$ for each $t \in [a, b]$.

The author does not know whether the conclusion of Lemma 1.47 would hold for E an arbitrary sequentially complete (FEP)-space and $\gamma \in \mathcal{L}^1([a, b], E)$.

1.48 If E is a Fréchet space or sequentially complete (FEP)-space and $\gamma \in \mathcal{L}^1([a, b], E)$ (resp., $\gamma \in \mathcal{L}_{rc}^\infty([a, b], E)$ with E an integral complete locally convex space), we define

$$\int_{t_0}^t [\gamma] := \int_{t_0}^t \gamma(s) ds$$

for $t, t_0 \in [a, b]$. The result is well-defined and only depends on $[\gamma]$. For $\gamma_1 \in [\gamma]$, we define $\int_{t_0}^t \gamma_1(s) ds := \int_{t_0}^t [\gamma] = \int_{t_0}^t \gamma(s) ds$.

1.49 Let E and F be real locally convex spaces, $U \subseteq E$ be open and $f: U \rightarrow F$ be a map. If f is continuous, we say that f is C^0 . We say that f is C^1 if f is continuous, the directional derivative

$$df(x, y) := (D_y f)(x) := \lim_{t \rightarrow 0} \frac{1}{t} (f(x + ty) - f(x))$$

(with $0 \neq t \in \mathbb{R}$) exists in F for all $(x, y) \in U \times E$, and $df: U \times E \rightarrow F$ is continuous. Recursively, for $k \in \mathbb{N}$ we say that f is C^k if f is C^1 and $df: U \times E \rightarrow F$ is C^{k-1} . We say that f is C^∞ (or smooth) if f is C^k for each $k \in \mathbb{N}_0$.

1.50 Let $r \in \mathbb{N}_0 \cup \{\infty\}$. It can be shown that a map $f: U \rightarrow F$ as before is C^r if and only if it is continuous, the iterated directional derivatives

$$d^{(k)} f(x, y_1, \dots, y_k) := (D_{y_k} \cdots D_{y_1} f)(x)$$

exist for all $k \in \mathbb{N}_0$ with $k \leq r$, $x \in U$ and $y_1, \dots, y_k \in E$, and the maps $d^{(k)} f: U \times E^k \rightarrow F$ so defined are continuous (see, e.g., [19] or [38]).

1.51 If E is a locally convex space and $U \subseteq E$ an open set, then

$$U^{[1]} := \{(x, y, t) \in U \times E \times \mathbb{R} : x + ty \in U\}$$

is an open subset of $E \times E \times \mathbb{R}$ which contains

$$U^{[1]1} := \{(x, y, t) \in U^{[1]} : t \neq 0\}$$

as an open dense subset. Moreover,

$$U^{[1]} = U^{[1]1} \cup (U \times E).$$

If $f: U \rightarrow F$ is a C^1 -map to a locally convex space F , then

$$f^{[1]}: U^{[1]} \rightarrow F, \quad (x, y, t) \mapsto \begin{cases} \frac{f(x+ty) - f(x)}{t} & \text{if } t \neq 0; \\ df(x, y) & \text{if } t = 0 \end{cases}$$

is continuous (see [5] or [38]). In two cases, we shall find this very useful.¹⁷

¹⁷Conversely, if there exists a continuous map $f^{[1]}: U^{[1]} \rightarrow F$ such that $f^{[1]}(x, y, t) = \frac{1}{t}(f(x + ty) - f(x))$ for all $(x, y, t) \in U^{[1]1}$, then f is C^1 (see [5] or [38]).

1.52 We shall use the Mean Value Theorem in integral form (see [38], [19]):

Let E and F be locally convex spaces, $U \subseteq E$ be an open subset, $f: U \rightarrow F$ a C^1 -map and $x, y \in U$ such that the line segment $\{tx + (1 - t)y: t \in [0, 1]\}$ is contained in U . Then

$$f(y) - f(x) = \int_0^1 df(x + sy, y) ds.$$

In particular, the preceding weak integral exists in F .

1.53 Since compositions of C^k -maps are C^k one can define C^k -manifolds and (smooth) Lie groups modelled on a real locally convex space E in the expected way, replacing the modelling space \mathbb{R}^n with E in the classical definitions of manifolds and Lie groups (see, e.g., [19] and [38] for streamlined expositions; cf. also [5], [52] and [54]). If we speak of manifolds, then these are modelled on a locally convex space (and thus need not be finite-dimensional). Likewise, the Lie groups we consider are modelled on arbitrary locally convex spaces. As usual, TM denotes the tangent bundle of a smooth manifold E modelled on a locally convex space, $T_x M$ the tangent space at $x \in M$, $\pi_{TM}: TM \rightarrow M$ the bundle projection sending $v \in T_x M$ to x , and $f: TM \rightarrow TN$ the tangent map of a smooth map $f: M \rightarrow N$ between manifolds. If U is an open subset of a locally convex space E , we identify TU with $U \times E$. If $f: M \rightarrow E$ is a smooth map from a smooth manifold to a locally convex space E , we write df for the second component of $Tf: TM \rightarrow TE = E \times E$.

1.54 As usual, an element $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is called a multi-index and $|\alpha| := \alpha_1 + \dots + \alpha_n$. Let $e_j \in \mathbb{R}^n$ be the standard basis vector with i -th component $\delta_{i,j}$ in terms of Kronecker's delta. If $U \subseteq \mathbb{R}^n$ is open, $k \in \mathbb{N}_0 \cup \{\infty\}$ and $f: U \rightarrow E$ a C^k -map to a locally convex space, we use the short-hand

$$\partial^\alpha f(x) := \frac{\partial^\alpha f}{\partial x^\alpha}(x) := ((D_{e_1})^{\alpha_1} \dots (D_{e_n})^{\alpha_n} f)(x)$$

for the partial derivatives of f at $x \in U$, for $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq k$.

We shall also use certain $C^{r,s}$ -maps $(x, y) \mapsto f(x, y)$ on products with different degrees of differentiability in x and y (see [1] and [2]):

Definition 1.55 Let E_1 , E_2 and F be real locally convex spaces, $U \subseteq E_1$ and $V \subseteq E_2$ be open subsets, $r, s \in \mathbb{N}_0 \cup \{\infty\}$ and $f: U \times V \rightarrow F$ be a map. Assume that the iterated directional derivatives

$$d^{(i,j)}f(x, w_1, \dots, w_i, y_1, \dots, y_j) := (D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,y_j)} \cdots D_{(0,y_1)}f)(x)$$

exist for all $i, j \in \mathbb{N}_0$ such that $i \leq r$ and $j \leq s$, and all $w_1, \dots, w_i \in E_1$ and $y_1, \dots, y_j \in E_2$. Moreover, assume that the mappings

$$d^{(i,j)}f: U \times V \times E_1^i \times E_2^j \rightarrow F$$

so obtained are continuous. Then f is called a $C^{r,s}$ -map.

$C^{r,s,t}$ -maps on products $U \times V \times W$ are defined analogously (see [1] for details). We shall use the following simple facts:

1.56 Let U , V , W , P and Q be open subsets of locally convex spaces, and E and F be locally convex spaces. Let $r, s, t, \tau \in \mathbb{N}_0 \cup \{\infty\}$. Then the following holds:

- (a) $f: U \times V \rightarrow F$ is C^∞ if and only if f is $C^{\infty,\infty}$ (see [2, Remark 3.16]).
- (b) If $f: E \times V \rightarrow F$ is $C^{(0,s)}$ for some $s \in \mathbb{N}_0 \cup \{\infty\}$ and $f(., y): E \rightarrow F$ is linear for each $y \in V$, then f is $C^{\infty,s}$ (cf. [2, Lemma 3.14]).
- (c) If $f: U \times V \rightarrow F$ is $C^{(r,s)}$, $g_1: P \rightarrow U$ is C^r and $g_2: Q \rightarrow V$ is C^s , then $g \circ (g_1 \times g_2): P \times Q \rightarrow F$ is $C^{(r,s)}$ (see [2, Lemma 3.17]).
- (d) If $g_1: P \times Q \rightarrow U$ is $C^{r,s}$, $g_2: W \rightarrow V$ a C^t -map and $f: U \times V \rightarrow F$ a C^τ -map with $\tau \geq r + s + t$, then $f \circ (g_1, g_2): P \times Q \times W \rightarrow F$ is $C^{r,s,t}$ (as a special case of [1, Lemma 81]).
- (e) If $f: U \times P \times Q \rightarrow F$ is $C^{r,k,\ell}$ for all $k, \ell \in \mathbb{N}_0 \cup \{\infty\}$ such that $k + \ell \leq s$, then $f: U \times (P \times Q) \rightarrow F$ is $C^{r,s}$ (see [1, Lemma 77]).
- (f) If $s < \infty$ and $s \geq 1$, then a map $f: U \times V \rightarrow F$ with V open in E is $C^{(r,s)}$ if and only if f is $C^{(r,0)}$ and $C^{(0,1)}$ and the partial differential $d_2f: U \times (V \times E) \rightarrow F$ is $C^{(r,s-1)}$ (see [2, Lemma 3.11]).

We need a version of the Chain Rule for curves in manifolds which are differentiable at a point.

Lemma 1.57 *Let E and F be locally convex spaces, $U \subseteq E$ be open and $f: U \rightarrow F$ be a C^1 -map. If $J \subseteq \mathbb{R}$ is a non-degenerate interval, $\eta: J \rightarrow U$ a continuous function and $t_0 \in J$ such that $\eta'(t_0)$ exists, then also $(f \circ \eta)'(t_0)$ exists, and $(f \circ \eta)'(t_0) = df(\eta(t_0), \eta'(t_0))$.*

1.58 Let M be a C^1 -manifold modelled on a locally convex space E , $J \subseteq \mathbb{R}$ be a non-degenerate interval and $t_0 \in J$. We say that a continuous curve $\eta: J \rightarrow M$ is *differentiable at t_0* if $(\phi \circ \gamma)'(t_0)$ exists in E for some chart $\phi: U \rightarrow V$ of M around $\gamma(t_0)$ (which takes the open neighbourhood U of $\gamma(t_0)$ in M diffeomorphically onto an open subset $V \subseteq E$). In this case, set

$$\gamma'(t_0) := T\phi^{-1}(\phi(\gamma(t_0)), (\phi \circ \gamma)'(t_0)) \in T_{\gamma(t_0)}M.$$

This definition is independent of the chosen chart.¹⁸

Lemma 1.59 *Let E and F be locally convex spaces, $V \subseteq E$ be open, $f: V \rightarrow F$ be a C^1 -map, $x \in V$ and $p \in P(F)$ be a continuous seminorm. Then there exists an open neighbourhood $V_1 \subseteq V$ of x and a continuous seminorm $q \in P(E)$ such that*

$$p(f(z) - f(y)) \leq q(z - x) \quad \text{for all } z, y \in V_1.$$

Now a simple compactness argument (cf. proof of Lemma 1.61) shows:

Lemma 1.60 *Let E and F be locally convex spaces, $V \subseteq E$ be open, $f: V \rightarrow F$ be a C^1 -map, $K \subseteq V$ a compact set and $p \in P(F)$ a continuous seminorm. Then there exists an open subset $V_1 \subseteq V$ with $K \subseteq V_1$ and a continuous seminorm $q \in P(E)$ such that*

$$p(f(z) - f(y)) \leq q(z - x) \quad \text{for all } z, y \in V_1.$$

Lemma 1.61 *Let E_1, E_2 and F be locally convex spaces, $V \subseteq E_1$ be open and $f: V \times E_2 \rightarrow F$ be a C^1 -map such that $f(x, \cdot): E_2 \rightarrow F$ is linear for all $x \in V$. Let $K \subseteq V$ be a compact set and $p \in P(F)$ be a continuous seminorm. Then there exist continuous seminorms $q_1 \in P(E_1)$ and $q_2 \in P(E_2)$ such that $K + B_1^{q_1}(0) \subseteq V$ and*

$$p(f(z, v) - f(y, w)) \leq q_2(v - w) + q_1(z - y)q_2(w) \quad (9)$$

for all $z, y \in K + B_1^{q_1}(0)$ and $v, w \in E_2$.

¹⁸If also ψ is a chart around $\gamma(t_0)$, then $\psi \circ \gamma = (\psi \circ \phi^{-1}) \circ (\phi \circ \gamma)$ on a neighbourhood of t_0 . By Lemma 1.57, $(\psi \circ \gamma)'(t_0)$ exists and equals $d(\psi \circ \phi)^{-1}(\phi(\gamma(t_0)), (\phi \circ \gamma)'(t_0))$. Hence $T\phi^{-1}(\phi(\gamma(t_0)), (\phi \circ \gamma)'(t_0)) = T\psi^{-1}T(\psi \circ \phi^{-1})(\phi(\gamma(t_0)), (\phi \circ \gamma)'(t_0)) = (\psi(\gamma(t_0)), (\psi \circ \gamma)'(t_0))$.

Note that, in contrast to Lemma 1.28, we cannot ensure differentiability almost everywhere in Lemma 1.29. The next lemma sometimes helps us to get around this difficulty. We use notation as in 1.45.

Lemma 1.62 *Let E and F be locally convex spaces, $V \subseteq E$ be open, $f: V \rightarrow F$ be a C^2 -map, $x \in V$ and $p \in P(F)$ be a continuous seminorm. Then there exists a continuous seminorm $q \in P(E)$ such that $B_1^q(x) \subseteq V$ and a C^1 -map*

$$g: B_1^{\|\cdot\|_q}(\pi_q(x)) \rightarrow \tilde{F}_p$$

on the ball $B_1^{\|\cdot\|_q}(\pi_q(x))$ in \tilde{E}_q such that

$$\pi_p \circ f|_{B_1^q(x)} = g \circ \pi_q|_{B_1^q(x)}.$$

Also the following generalization of Lemma 1.62 will be used, which follows by a simple compactness argument.

Lemma 1.63 *Let E and F be locally convex spaces, $V \subseteq E$ be open, $f: V \rightarrow F$ be a C^2 -map, $K \subseteq V$ a compact set and $p \in P(F)$ be a continuous seminorm. Then there exists an open set $V_1 \subseteq V$ with $K \subseteq V_1$, a continuous linear map $\lambda: E \rightarrow Y$ to a Banach space Y and a C^1 -map $g: W \rightarrow \tilde{F}_q$ on an open subset $W \subseteq Y$ with $\lambda(V_1) \subseteq W$ such that*

$$\pi_q \circ f|_{V_1} = g \circ \lambda|_{V_1}^W.$$

1.64 Recall that a mapping $p: E \rightarrow F$ between complex locally convex spaces is called a *continuous homogeneous polynomial* of degree $n \in \mathbb{N}_0$ if there exists a continuous complex n -linear map $\beta: E^n \rightarrow F$ such that

$$p(x) = \beta(x, x, \dots, x) \quad \text{for all } x \in E.$$

In some cases, we can even establish analyticity (rather than mere smoothness) of evolution maps. Analyticity is understood in the following sense.

1.65 Let E and F be complex locally convex spaces and $U \subseteq E$ be an open set. Following [7] (see also [19] and [38]), a mapping $f: U \rightarrow F$ is called *complex analytic* if f is continuous and each $z \in U$ has an open neighbourhood $W \subseteq U$ such that

$$f(w) = \sum_{n=0}^{\infty} p_n(w - z) \quad \text{for all } w \in W,$$

with pointwise convergence, for some sequence of continuous homogeneous polynomials $p_n: E \rightarrow F$ of degree n .

1.66 For E, F, U as in 1.65, a map $f: U \rightarrow F$ is complex analytic if and only if f is C^∞ on the underlying real locally convex spaces, with complex linear differentials

$$df(x, \cdot): E \rightarrow F$$

for all $x \in U$ (see [19]). If F is integral complete (or only Mackey complete), then complex analyticity of f follows if f is C^1 with complex linear differentials [38].

If E is a real locally convex space, then the direct product $E_{\mathbb{C}} := E \times E$ becomes a complex locally convex space if we define $(x + iy)(v, w) := (xv - yw, yv + xw)$ for $x, y \in \mathbb{R}, v, w \in E$. Identifying E with $E \times \{0\} \subseteq E_{\mathbb{C}}$, we have $E_{\mathbb{C}} = E \oplus iE$ and $(x + iy)(v + iw) = (xv - yw) + i(yv + xw)$.

Real analytic maps are defined via existence of complex analytic extensions.

1.67 Let E and F be real locally convex spaces and $U \subseteq E$ be an open set. Following [19] and [5] (cf. also [52]), a map $f: U \rightarrow F$ is called *real analytic* if there exists a complex analytic map

$$\tilde{f}: \tilde{U} \rightarrow F_{\mathbb{C}}$$

on an open subset $\tilde{U} \subseteq E_{\mathbb{C}}$ with $U \subseteq \tilde{U}$, such that $f = \tilde{f}|_U$.

Compositions of complex analytic maps are complex analytic; compositions of real analytic mappings are real analytic (see, e.g., [19] or [38]).

2 Mappings between Lebesgue spaces

We establish continuity and differentiability properties for certain non-linear maps between Lebesgue spaces as considered in Section 1, with parameters.

Lemma 2.1 *Let E_1, E_2 and F be locally convex spaces, $V \subseteq E_1$ be an open set and $f: V \times E_2 \rightarrow F$ be a continuous map such that $f(x, \cdot): E_2 \rightarrow F$ is linear for each $x \in V$. Assume the following:*

- (a) E_2 and F are integral complete and \mathcal{E} denotes \mathcal{L}_{rc}^∞ or \mathcal{R} ; or
- (b) E_2 and F are Fréchet spaces or (FEP)-spaces and \mathcal{E} denotes \mathcal{L}^p for some $p \in [1, \infty]$.

Let $a, b \in \mathbb{R}$ such that $a < b$, $\eta: [a, b] \rightarrow V$ be a continuous function, and $\gamma \in \mathcal{E}([a, b], E_2)$. Then

$$\theta := f \circ (\eta, \gamma) \in \mathcal{E}([a, b], F).$$

Proposition 2.2 *If E_2 and F are integral complete locally convex spaces in the situation of Lemma 2.1, \mathcal{E} is \mathcal{L}_{rc}^∞ or \mathcal{R} and the map f is C^k for some $k \in \mathbb{N}_0 \cup \{\infty\}$, then also the map*

$$\tilde{f}: C([a, b], V) \times \mathcal{E}([a, b], E) \rightarrow \mathcal{E}([a, b], F), \quad (\eta, [\gamma]) \mapsto [f \circ (\eta, \gamma)] \quad (10)$$

is C^k .

Proposition 2.3 *If E_2 and F are Fréchet spaces or (FEP)-spaces in the situation of Lemma 2.1, $\mathcal{E} = L^p$ with $p \in [1, \infty]$ and f is C^{k+1} for some $k \in \mathbb{N}_0 \cup \{\infty\}$, then the map*

$$\tilde{f}: C([a, b], V) \times \mathcal{E}([a, b], E) \rightarrow \mathcal{E}([a, b], F), \quad (\eta, [\gamma]) \mapsto [f \circ (\eta, \gamma)] \quad (11)$$

is C^k .

Proof of Lemma 2.1. Since $\eta([a, b])$ metrizable and compact and hence second countable, we have $\mathcal{B}(\eta([a, b]) \times E_2) = \mathcal{B}(\eta([a, b])) \otimes \mathcal{B}(E_2)$ (see 1.6 (f)). As f is continuous and hence Borel measurable, we deduce that $f \circ (\eta, \gamma)$ is measurable. Let $D_1 \subseteq \eta([a, b])$ and $D_2 \subseteq \gamma([a, b])$ be dense countable subsets. Since f is continuous, the countable set $f(D_1 \times D_2)$ is dense in $\text{im}(\theta)$. Hence $\theta: [a, b] \rightarrow F$ is measurable and has separable image. Let $q \in P(F)$ be a continuous seminorm. The map

$$h: \eta([a, b]) \times E_2 \rightarrow F, \quad h(t, v) := f(\eta(t), v)$$

is continuous and $h([a, b] \times \{0\}) = 0$. Using the Wallace Lemma (see 1.1), we find an open subset $V_1 \subseteq V$ such that $\eta([a, b]) \subseteq V_1$ and an open 0-neighbourhood $W \subseteq E_2$ such that $h(V_1 \times W) \subseteq B_1^q(0)$. We find a continuous seminorm $Q \in P(E_2)$ such that $B_1^Q(0) \subseteq W$, and we may assume that

$V_1 = \eta([a, b]) + B_1^P(0)$ for a continuous seminorm P on E_1 . Thus $q(v, w) \leq 1$ for all $v \in V_1$ and $w \in B_1^Q(0)$ and hence

$$q(f(v, w)) \leq Q(w) \quad \text{for all } v \in V_1 \text{ and } w \in E_2. \quad (12)$$

If $\mathcal{E} = \mathcal{L}^p$ with $p \in [1, \infty[$, we have $q(\theta(t)) = q(f(\eta(t), \gamma(t))) \leq Q(\gamma(t))$ and thus

$$\|\theta\|_{\mathcal{L}^p, q} = \sqrt[p]{\int_a^b q(\theta(t))^p dt} \leq \sqrt[p]{\int_a^b Q(\gamma(t))^p dt} = \|\gamma\|_{\mathcal{L}^p, Q} < \infty.$$

Hence $\theta \in \mathcal{L}^p([a, b], F)$.

If $\mathcal{E} = \mathcal{L}^\infty$, we have $q(\theta(t)) \leq Q(\gamma(t))$ and thus $\sup q(\theta([a, b])) \leq \sup Q(\gamma([a, b])) < \infty$, entailing that $\theta([a, b]) \subseteq F$ is bounded. Thus $\theta \in \mathcal{L}^\infty([a, b], F)$. Moreover, $\|\theta\|_{\mathcal{L}^\infty, q} \leq \|\gamma\|_{\mathcal{L}^\infty, Q}$ by the preceding.

If $\mathcal{E} = \mathcal{L}_{rc}^\infty$, then the set $f(\eta([0, 1])) \times \overline{\text{im}(\gamma)}$ is compact and metrizable (see Lemma 1.10), entailing that $\theta \in \mathcal{L}_{rc}^\infty([a, b], F)$.

If $\mathcal{E} = \mathcal{R}$, we choose $\gamma_n \in \mathcal{T}([a, b], E_2)$ such that $\gamma_n \rightarrow \gamma$ uniformly. Since $C([a, b], E_1) \subseteq \mathcal{R}([a, b], E_1)$, we also find $\eta_n \in \mathcal{T}([a, b], E_1)$ such that $\eta_n \rightarrow \eta$ uniformly. Then $(\eta_n, \gamma_n) \in \mathcal{T}([a, b], E_1 \times E_2)$. There is a continuous seminorm $P \in P(E_1)$ such that $\eta([a, b]) + B_1^P(0) \subseteq V$. After passing to a subsequence, we may assume that $\sup_{t \in [a, b]} P(\eta(t) - \eta_n(t)) < 1$, whence $\eta_n(t) \in V$ for all $t \in [0, 1]$ and thus

$$f \circ (\eta_n, \gamma_n) \in \mathcal{T}([a, b], F)$$

for all $n \in \mathbb{N}$. Given $q \in P(F)$, we choose $Q \in P(E_2)$ as above. Let $K := \overline{\text{im}(\gamma)}$. We consider the continuous function

$$g: [a, b] \times E_2 \times B_1^P(0) \rightarrow F, \quad g(t, y, v) := f(\eta(t) + v, y) - f(\eta(t), y).$$

Since $g([a, b] \times K \times \{0\}) = \{0\} \subseteq B_1^q(0)$, the Wallace Lemma implies that there is $S \in P(E_1)$ with $S \geq P$ such that $g([a, b] \times K \times B_1^S(0)) \subseteq B_1^q(0)$. We find $N \in \mathbb{N}$ such that $\sup_{t \in [a, b]} Q(\gamma(t) - \gamma_n(t)) < 1$ and $\sup_{t \in [a, b]} R(\eta(t) - \eta_n(t)) < 1$ for all $n \geq N$. For $n \geq N$ and $t \in [a, b]$, we obtain

$$\begin{aligned} & q(f(\eta(t), \gamma(t)) - f(\eta_n(t), \gamma_n(t))) \\ & \leq q(f(\eta(t), \gamma(t)) - f(\eta_n(t), \gamma(t)) + q(f(\eta_n(t), \gamma(t) - \gamma_n(t))) \leq 2, \end{aligned}$$

showing that $f \circ (\eta_n, \gamma_n) \rightarrow f \circ (\eta, \gamma) = \theta$ uniformly. Thus $\theta \in \mathcal{R}([a, b], F)$. \square

Proof of Propositions 2.2 and 2.3. To see that \tilde{f} is C^k if f is C^k (resp., C^{k+1}), we may assume that $k \in \mathbb{N}_0$ and proceed by induction. The case $k = 0$ is a special case of the following lemma. The induction step will be completed once Lemma 2.4 is available.

Lemma 2.4 *Let E_1 , E_2 and F be locally convex spaces, $V \subseteq E_1$ be an open set, Λ be a topological space, $a, b \in \mathbb{R}$ such that $a < b$ and*

$$f: \Lambda \times V \times E_2 \rightarrow F$$

be a map such that $f(\lambda, x, \cdot): E_2 \rightarrow F$ is linear for all $(\lambda, x) \in \Lambda \times V$. Assume the following:

- (a) *E_2 and F are integral complete, f is continuous and \mathcal{E} denotes \mathcal{L}_{rc}^∞ or \mathcal{R} ; or*
- (b) *E_2 and F are Fréchet spaces or (FEP)-spaces, Λ is an open subset of a locally convex space E_0 , the map f is C^1 and \mathcal{E} denotes \mathcal{L}^p for some $p \in [1, \infty]$.*

Then the map

$$\tilde{f}: \Lambda \times C([a, b], V) \times \mathcal{E}([a, b], E) \rightarrow \mathcal{E}([a, b], F), \quad (\lambda, \eta, \gamma) \mapsto f(\lambda, \cdot) \circ (\eta, \gamma) \quad (13)$$

is continuous.

Proof. Let us show that \tilde{f} is continuous at each (λ, η, γ) .

Let $q \in P(F)$ be a continuous seminorm. If $\mathcal{E} = \mathcal{L}_{rc}^\infty$ or $\mathcal{E} = \mathcal{R}$, we proceed as follows: The map

$$h: \Lambda \times \eta([a, b]) \times E_2 \rightarrow F, \quad h(t, v) := f(\eta(t), v)$$

is continuous and $h(\Lambda \times [a, b] \times \{0\}) = 0$. Using the Wallace Lemma (see 1.1), we find an open neighbourhood $V_0 \subseteq \Lambda$ of λ , an open 0-neighbourhood $W \subseteq E$ and an open set $V_1 \subseteq V$ such that $\eta([a, b]) \subseteq V_1$ and $h(V_0 \times V_1 \times W) \subseteq B_1^q(0)$. We find a continuous seminorm $Q \in P(E_2)$ such that $B_1^Q(0) \subseteq W$, and we may assume that $V_1 = \eta([a, b]) + B_1^P(0)$ for a continuous seminorm P on E_1 . Thus $q(\mu, v, w) \leq 1$ for all $\mu \in V_0$, $v \in V_1$ and $w \in B_1^Q(0)$ and hence

$$q(f(\mu, v, w)) \leq Q(w) \quad \text{for all } \mu \in V_0, v \in V_1 \text{ and } w \in E_2. \quad (14)$$

Let $K := \overline{\text{im}(\gamma)}$. We consider the continuous function

$$g: V_0 \times [a, b] \times E_2 \times B_1^P(0) \rightarrow F, \quad g(\mu, t, y, v) := f(\mu, \eta(t) + v, y) - f(\lambda, \eta(t), y).$$

Since $g(V_0 \times [a, b] \times K \times \{0\}) = \{0\} \subseteq B_1^q(0)$, the Wallace Lemma implies that, after shrinking V_0 if necessary, there is $S \in P(E_1)$ with $S \geq P$ such that $g(V_0 \times [a, b] \times K \times B_1^S(0)) \subseteq B_1^q(0)$. Then

$$\begin{aligned} & q(f(\bar{\lambda}, \bar{\eta}(t), \bar{\gamma}(t)) - f(\lambda, \eta(t), \gamma(t))) \\ & \leq q(f(\bar{\lambda}, \bar{\eta}(t), \bar{\gamma}(t) - \gamma(t))) + q(f(\bar{\lambda}, \bar{\eta}(t), \gamma(t)) - f(\lambda, \eta(t), \gamma(t))) \\ & \leq Q(\bar{\gamma}(t) - \gamma(t)) + 1 \leq 2 \end{aligned}$$

for λ_1 -almost all $t \in [a, b]$, for all $\bar{\lambda} \in V_0$, $\bar{\eta} \in \eta + C([a, b], B_1^P(0))$ and $\bar{\gamma} \in \gamma + \Omega$ with the 0-neighbourhood $\Omega \subseteq \mathcal{E}([a, b], E)$ consisting of all $\zeta \in \mathcal{E}([a, b], E)$ such that $\zeta(t) \in B_1^Q(0) \cap B_1^S(0)$ for λ_1 -almost all $t \in [a, b]$. Hence f is continuous at (λ, η, γ) .

In the situation of (b), we have $\mathcal{E} = \mathcal{L}^p$ with $p \in [1, \infty]$. Moreover, f is C^1 . We apply Lemma 1.61 with $K := \{\lambda\} \times \eta([a, b]) \subseteq \Lambda \times V \subseteq E_0 \times E_1$. Given a continuous seminorm $q \in P(F)$, it provides continuous seminorms $Q \in P(E_0 \times E_1)$ and $q_2 \in P(E_2)$ such that $K + B_1^Q(0) \subseteq \Lambda \times V$ and

$$q(f(\sigma, z, v) - f(\tau, y, w)) \leq q_2(v - w) + Q((\sigma, z) - (\tau, y))q_2(w)$$

for all $(\sigma, y), (\tau, z) \in K + B_1^Q(0)$ and $v, w \in E_2$. After increasing Q , we may assume that $Q(\sigma, z) = \max\{q_0(\sigma), q_1(z)\}$ for continuous seminorms $q_0 \in P(E_0)$ and $q_1 \in P(E_1)$. Thus

$$q(f(\sigma, z, v) - f(\tau, y, w)) \leq q_2(v - w) + \max\{q_0(\sigma - \tau), q_1(z - y)\}q_2(w)$$

for all $\sigma, \tau \in B_1^{q_0}(\lambda)$, $y, z \in \text{im}(\eta) + B_1^{q_1}(0)$ and $v, w \in E_2$. Hence, for all $\bar{\lambda} \in B_1^{q_0}(\lambda)$, $\bar{\eta} \in \eta + C([a, b], B_1^{q_1}(0))$ and $\gamma, \bar{\gamma}$ in $\mathcal{L}^p([a, b], E_2)$ we have

$$\begin{aligned} & q(f(\bar{\lambda}\bar{\eta}(t), \bar{\gamma}(t)) - f(\lambda, \eta(t), \gamma(t))) \\ & \leq q_2(\bar{\gamma}(t) - \gamma(t)) + \max\{q_0(\bar{\lambda} - \lambda), q_1(\bar{\eta}(t) - \eta(t))\}q_2(\gamma(t)) \end{aligned}$$

for all $t \in [a, b]$. If $p = \infty$, we deduce that

$$\|\tilde{f}(\bar{\lambda}, \bar{\eta}, \bar{\gamma}) - \tilde{f}(\lambda, \eta, \gamma)\|_{\mathcal{L}^\infty, q} \|\bar{\gamma} - \gamma\|_{\mathcal{L}^\infty, q_2} + \max\{q_0(\bar{\lambda} - \lambda), \|\bar{\eta} - \eta\|_{\mathcal{L}^\infty, q_1}\} \|\gamma\|_{\mathcal{L}^\infty, q_2},$$

which tends to 0 if $(\bar{\lambda}, \bar{\eta}, \bar{\gamma}) \rightarrow (\lambda, \eta, \gamma)$. If $p \in [1, \infty[$, we deduce that

$$\begin{aligned} & \|\tilde{f}(\bar{\lambda}, \bar{\eta}, \bar{\gamma}) - \tilde{f}(\lambda, \eta, \gamma)\|_{\mathcal{L}^p, q} \\ &= \|q \circ (\tilde{f}(\bar{\lambda}, \bar{\eta}, \bar{\gamma}) - \tilde{f}(\lambda, \eta, \gamma))\|_{\mathcal{L}^p} \\ &\leq \|q_2 \circ (\bar{\gamma} - \gamma)\|_{\mathcal{L}^p} + \|\max\{q_0(\bar{\lambda} - \lambda), (q_1 \circ (\bar{\eta} - \eta))\} \cdot (q_2 \circ \gamma)\|_{\mathcal{L}^p} \\ &\leq \|\bar{\gamma} - \gamma\|_{\mathcal{L}^p, q_2} + \max\{q_0(\bar{\lambda} - \lambda), \|\bar{\eta} - \eta\|_{\mathcal{L}^\infty, q_1}\} \|\gamma\|_{\mathcal{L}^p, q_2}. \end{aligned}$$

As the right hand side tends to 0 as $(\bar{\lambda}, \bar{\eta}, \bar{\gamma}) \rightarrow (\lambda, \eta, \gamma)$ in $C([a, b], E_1) \times \mathcal{L}^p([a, b], E_2)$, we see that \tilde{f} is continuous at (λ, η, γ) . \square

Proof of Propositions 2.2 and 2.3, completed. Let $k \in \mathbb{N}_0$ and assume that the assertion holds for k (i.e., \tilde{f} is C^k). If \mathcal{E} is L_{rc}^∞ or R , assume that f is C^{k+1} ; if $\mathcal{E} = L^p$ for some $p \in [1, \infty]$, assume that f is C^{k+2} . We show that \tilde{f} is C^{k+1} with

$$d\tilde{f}(\eta, [\gamma], \eta_1, [\gamma_1]) = [df \circ (\eta, \gamma, \eta_1, \gamma_1)] \quad (15)$$

for all $\eta \in C([a, b], V)$, $\eta_1 \in C([a, b], E_1)$ and $[\gamma], [\gamma_1] \in \mathcal{E}([a, b], E_2)$. Consider the open set

$$(V \times E_2)^{[1]}$$

$$:= \{(x, v, y, w, t) \in V \times E_2 \times E_1 \times E_2 \times \mathbb{R} : (x + ty, v + tw) \in V \times E_2\}$$

in $E_1 \times E_2 \times E_1 \times E_2 \times \mathbb{R}$ (cf. 1.51). By 1.51, there is a unique continuous map

$$f^{[1]}: (V \times E_2)^{[1]} \rightarrow F$$

such that

$$f^{[1]}(x, v, y, w, t) = \frac{f(x + ty, v + tw) - f(x, v)}{t}$$

for all $(x, v, y, w, t) \in (V \times E_2)^{[1]}$ such that $t \neq 0$. Then $f^{[1]}(x, v, y, w, 0) = df((x, v), (y, w))$ for all $(x, y) \in V \times E_1$. If f is C^{k+1} , then $f^{[1]}$ is C^k ; if f is C^{k+2} , then $f^{[1]}$ is C^{k+1} (see [5]). Since $\eta_1([a, b])$ is compact, there exists an open 0-neighbourhood $U \subseteq E_1$ such that $\eta_1([a, b]) + U \subseteq V$. Choose an open balanced 0-neighbourhood $W \subseteq E_1$ such that $W + W \subseteq U$. There is $\varepsilon > 0$ such that $\text{im}(\eta_1) \subseteq \varepsilon^{-1}W$. Then

$$(\eta_1([a, b] + W) \times E_2 \times \varepsilon^{-1}W \times E_2 \times]-\varepsilon, \varepsilon[\subseteq (V \times E_2)^{[1]}.$$

Consider the function $g:]-\varepsilon, \varepsilon[\times ((\eta([a, b]) + W) \times \varepsilon^{-1}W) \times (E_2 \times E_2) \rightarrow F$,

$$g(t, (x, y), (v, w)) := f^{[1]}(x, v, y, w, t).$$

Lemma 2.4 entails that the function

$$\tilde{g}:]-\varepsilon, \varepsilon[\times C([a, b], (\eta([a, b] + W) \times \varepsilon^{-1}W) \times \mathcal{E}([a, b], E_2 \times E_2) \rightarrow \mathcal{E}([a, b], F),$$

$$\tilde{g}(t, (\bar{\eta}, \bar{\eta}_1), [(\bar{\gamma}, \bar{\gamma}_1)]) := [g(t, \cdot) \circ (\bar{\eta}, \bar{\eta}_1, \bar{\gamma}, \bar{\gamma}_1)] = [f^{[1]}(\cdot, t) \circ (\bar{\eta}, \bar{\gamma}, \bar{\eta}_1, \bar{\gamma}_1)]$$

is continuous. Therefore the map

$$]-\varepsilon, \varepsilon[\rightarrow \mathcal{E}([a, b], F), \quad t \mapsto \Delta_t := \tilde{g}(t, \eta, \eta_1, [(\gamma, \gamma_1)])$$

is continuous. Letting $t \in]-\varepsilon, \varepsilon[\setminus \{0\}$ tend to 0, we deduce that

$$\frac{[f \circ (\eta + t\eta_1, \gamma + t\gamma_1)] - [f \circ (\eta, \gamma)]}{t} = \Delta_t \rightarrow \Delta_0 = [df \circ (\eta, \gamma, \eta_1, \gamma_1)].$$

Thus $d\tilde{f}((\eta, [\gamma]), (\eta_1, [\gamma_1]))$ exists and is given by

$$d\tilde{f}((\eta, [\gamma]), (\eta_1, [\gamma_1])) = df \circ (\eta, [\gamma], \eta_1, [\gamma_1]) = \tilde{h}((\eta, \eta_1), [(\gamma, \gamma_1)]) \quad (16)$$

with $h: (V \times E_1) \times (E_2 \times E_2) \rightarrow F$, $h((x, y), (v, w)) := df((x, v), (y, w))$. Note that, for fixed $(x, y, t) \in V^{[1]}$ with $t \neq 0$,

$$f^{[1]}((x, v), (y, w), t) = \frac{f(x + ty, v + tw) - f(x, v)}{t}$$

is linear in (v, w) . Letting $t \rightarrow 0$, we see that also

$$h((x, y), (v, w)) = df((x, v), (y, w)) = f^{[0]}((x, v), (y, w), 0)$$

is linear in $(v, w) \in E_2 \times E_2$ for fixed $(x, y) \in V \times E_1$. Note that h is C^k if \mathcal{E} is L_{rc}^∞ or \mathcal{R} ; if \mathcal{E} is L^p , then h is C^{k+1} . Hence \tilde{h} is C^k by induction, and hence $d\tilde{f}$ is C^k , by (16). In particular, $d\tilde{f}$ is continuous, and thus \tilde{f} is C^1 . Since \tilde{f} is C^1 and $d\tilde{f}$ is C^k , the map \tilde{f} is C^{k+1} . \square

3 The spaces $AC_{\mathcal{E}}([a, b], E)$ and mappings between them

In this section, we define spaces of absolutely continuous functions $\eta: [a, b] \rightarrow E$ with values in integral complete locally convex spaces E . For general E , we wish to distinguish the cases that $\eta' \in L_{rc}^\infty([a, b], E)$ and $\eta' \in R([a, b], E)$

(a regulated function), respectively. And if E is a Fréchet space (or a sequentially complete (FEP)-space), we also wish to distinguish the cases that $\eta' \in L^p([a, b], E)$ for some $p \in [1, \infty]$. To enable a uniform treatment of all of these cases, we find it convenient to assume that locally convex spaces

$$\mathcal{E}([a, b], E) \subseteq L_{rc}^\infty([a, b], E)$$

(resp., $\mathcal{E}([a, b], E) \subseteq L^1([a, b], E)$) have been selected for all $a, b \in \mathbb{R}$ such that $a < b$ and all integral complete locally convex spaces E (resp., all Fréchet spaces E , resp., all sequentially complete (FEP)-spaces) in a reasonable way; we then speak of a *bifunctor* on integral complete locally convex spaces (resp., on Fréchet spaces, resp., on sequentially complete (FEP)-spaces). Given such a bifunctor, we define and study certain locally convex spaces $AC_{\mathcal{E}}([a, b], E)$ of absolutely continuous functions $\eta: [a, b] \rightarrow E$ with $\eta' \in \mathcal{E}([a, b], E)$. Natural additional axioms are worked out which enable $AC_{\mathcal{E}}([a, b], M)$ to be defined also for M a manifold modelled on E ; they are satisfied by all of L^p , L_{rc}^∞ , and R . Later, we shall associate a Lie group $AC_{\mathcal{E}}([0, 1], G)$ to each Lie group G modelled on E . The Lie group $AC_{\mathcal{E}}([a, b], G)$ is needed to define the notion of \mathcal{E} -regularity for the Lie group G .

Definition 3.1 Assume that, for each Fréchet space E and $a, b \in \mathbb{R}$ such that $a < b$, a vector subspace $\mathcal{E}([a, b], E)$ of $L^1([a, b], E)$ has been assigned, together with a locally convex vector topology on $\mathcal{E}([a, b], E)$ such that the inclusion map

$$\mathcal{E}([a, b], E) \rightarrow L^1([a, b], E), \quad [\gamma] \mapsto [\gamma]$$

is continuous. Consider the following conditions:

- (B1) For each continuous linear map $\lambda: E_1 \rightarrow E_2$ between Fréchet spaces, we have $[\lambda \circ \gamma] \in \mathcal{E}([a, b], E_2)$ for all $a, b \in \mathbb{R}$ such that $a < b$ and $[\gamma] \in \mathcal{E}([a, b], E_1)$, and the linear map

$$\mathcal{E}([a, b], \lambda): \mathcal{E}([a, b], E_1) \rightarrow \mathcal{E}([a, b], E_2), \quad [\gamma] \mapsto [\lambda \circ \gamma]$$

is continuous.

- (B2) For each Fréchet space E , real numbers $a, b, \alpha, \beta, c, d$ with $a \leq \alpha < \beta \leq b$ and $c < d$, consider the map $f: [c, d] \rightarrow [a, b]$ given by

$$f(t) := \alpha + \frac{t - c}{d - c}(\beta - \alpha) \quad \text{for } t \in [c, d]. \quad (17)$$

We require that $[\gamma \circ f] \in \mathcal{E}([c, d], E)$ for each $[\gamma] \in \mathcal{E}([a, b], E)$ and that the linear map

$$\mathcal{E}(f, E): \mathcal{E}([a, b], E) \rightarrow \mathcal{E}([c, d], E), \quad [\gamma] \mapsto [\gamma \circ f]$$

is continuous.¹⁹

We call \mathcal{E} a *bifunctor* on Fréchet spaces if (B1) and (B2) are satisfied.

Remark 3.2 In particular, (B2) requires that the linear map

$$\rho: \mathcal{E}([a, b], E) \rightarrow \mathcal{E}([\alpha, \beta], E), \quad [\gamma] \mapsto [\gamma|_{[\alpha, \beta]}]$$

is continuous, for all $a \leq \alpha < \beta \leq b$.

3.3 If Fréchet spaces are replaced with sequentially complete (FEP)-spaces in Definition 3.1, then we speak of a *bifunctor on sequentially complete (FEP)-spaces*.

Definition 3.4 Assume that, for each integral complete locally convex space E and $a, b \in \mathbb{R}$ such that $a < b$, a vector subspace $\mathcal{E}([a, b], E)$ of $L_{rc}^\infty([a, b], E)$ has been assigned, together with a locally convex vector topology on $\mathcal{E}([a, b], E)$ such that the inclusion map $\mathcal{E}([a, b], E) \rightarrow L_{rc}^\infty([a, b], E)$ is continuous. Consider the following conditions:

(B1) For each continuous linear map $\lambda: E_1 \rightarrow E_2$ between integral complete locally convex spaces, we have $[\lambda \circ \gamma] \in \mathcal{E}([a, b], E_2)$ for all $a, b \in \mathbb{R}$ such that $a < b$ and $[\gamma] \in \mathcal{E}([a, b], E_1)$, and the linear map

$$\mathcal{E}([a, b], \lambda): \mathcal{E}([a, b], E_1) \rightarrow \mathcal{E}([a, b], E_2), \quad [\gamma] \mapsto [\lambda \circ \gamma]$$

is continuous.

(B2) For each integral complete locally convex space E , real numbers $a, b, \alpha, \beta, c, d$ with $a \leq \alpha < \beta \leq b$ and $c < d$, let $f: [c, d] \rightarrow [a, b]$ be as in (17). We require that $[\gamma \circ f] \in \mathcal{E}([c, d], E)$ for each $[\gamma] \in \mathcal{E}([a, b], E)$ and that the linear map

$$\mathcal{E}(f, E): \mathcal{E}([a, b], E) \rightarrow \mathcal{E}([c, d], E), \quad [\gamma] \mapsto [\gamma \circ f]$$

is continuous.

¹⁹In other words, we can pull back along affine-linear maps.

We call \mathcal{E} a *bifunctor* on integral complete locally convex spaces if (B1) and (B2) are satisfied.

Remark 3.5 The condition (B1) entails that

$$\mathcal{E}([a, b], E_1 \times E_2) \cong \mathcal{E}([a, b], E_1) \times \mathcal{E}([a, b], E_2)$$

as locally convex spaces, for all $a, b \in \mathbb{R}$ such that $a < b$ and Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) E_1 and E_2 . To see this, let $\pi_j : E_1 \times E_2 \rightarrow E_j$ be the projection onto the j -th component, for $j \in \{1, 2\}$. We then see as in the proof of 1.35 that $(\mathcal{E}([a, b], \pi_1), \mathcal{E}([a, b], \pi_2))$ is an isomorphism of locally convex spaces.

Definition 3.6 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., a bifunctor on sequentially complete (FEP)-spaces, resp., a bifunctor on integral complete locally convex spaces). Let $a < b$ be real numbers and E be a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space). Let $t_0 \in [a, b]$. We define $AC_{\mathcal{E}}([a, b], E) \subseteq C([a, b], E)$ as the space of all continuous functions $\eta : [a, b] \rightarrow E$ for which there exists a $[\gamma] \in \mathcal{E}([a, b], E)$ such that

$$(\forall t \in [a, b]) \quad \eta(t) = \eta(t_0) + \int_{t_0}^t \gamma(s) ds.$$

Lemma 1.28 (resp., Lemma 1.29) implies that $[\gamma] = \eta'$ is unique, and that the map

$$\Phi : AC_{\mathcal{E}}([a, b], E) \rightarrow E \times \mathcal{E}([a, b], E), \quad \eta \mapsto (\eta(t_0), \eta')$$

is an isomorphism of vector spaces (with $\Phi^{-1}(v, [\gamma])(t) := v + \int_{t_0}^t \gamma(s) ds$). We give $AC_{\mathcal{E}}([a, b], E)$ the Hausdorff locally convex vector topology which makes Φ an isomorphism of topological vector spaces.

We shall see in Remark 3.10 that both the definition of $AC_{\mathcal{E}}([a, b], E)$ and its topology are independent of the choice of t_0 .

Remark 3.7 (a) All of L^p for $p \in [1, \infty]$ define bifunctors on Fréchet spaces, as well as L_{rc}^{∞} and R . Therefore, we obtain function spaces

$AC_{L^p}([a, b], E)$ with $p \in [1, \infty]$; $AC_{L_{rc}^\infty}([a, b], E)$, and $AC_R([a, b], E)$ such that

$$\begin{aligned} AC_R([a, b], E) &\subseteq AC_{L_{rc}^\infty}([a, b], E) \subseteq AC_{L^\infty}([a, b], E) \\ &\subseteq AC_{L^p}([a, b], E) \subseteq AC_{L^p}([a, b], E) \subseteq AC_{L^1}([a, b], E) \end{aligned}$$

with continuous inclusion maps, whenever $\infty \geq p \geq q \geq 1$. Likewise for Fréchet spaces replaced with sequentially complete (FEP)-spaces.

(b) L_{rc}^∞ and R define bifunctors on integral complete locally convex spaces.

Remark 3.8 If $\eta: [a, b] \rightarrow \mathbb{R}$, then $\eta(t) = \eta(a) + \int_a^t \gamma(s) ds$ for some $\gamma \in \mathcal{L}^1([0, 1], \mathbb{R})$ if and only if η is absolutely continuous in the sense that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{j=1}^n |\eta(b_j) - \eta(a_j)| < \varepsilon$$

for all $n \in \mathbb{N}$ and disjoint subintervals $]a_1, b_1[, \dots,]a_n, b_n[$ of $[a, b]$ of total length $\sum_{j=1}^n (b_j - a_j) < \delta$.

[In fact, if η is absolutely continuous in the latter sense, then η is a primitive of an \mathcal{L}^1 -function by [60, Theorem 7.20]. If, conversely, $\eta(t) = \eta(a) + \int_a^t \gamma(s) ds$ for some $\gamma \in \mathcal{L}^1([a, b], \mathbb{R})$, consider the auxiliary function

$$\zeta: [a, b] \rightarrow \mathbb{R}, \quad \zeta(t) := \int_a^t |\gamma(s)| ds.$$

Then ζ is differentiable λ_1 -almost everywhere and $\zeta'(t)$ coincides with $\gamma(t)$ for λ_1 -almost every $t \in [a, b]$ (cf. [60, Theorem 7.11]). Now (c) \Rightarrow (a) in [60, Theorem 7.18] shows that ζ is absolutely continuous. Hence, given $\varepsilon > 0$, we find $\delta > 0$ such that

$$\sum_{j=1}^n |\zeta(b_j) - \zeta(a_j)| < \varepsilon$$

for all $n \in \mathbb{N}$ and disjoint subintervals $]a_1, b_1[, \dots,]a_n, b_n[$ of $[a, b]$ of total length $\sum_{j=1}^n (b_j - a_j) < \delta$. Since

$$|\eta(b_j) - \eta(a_j)| = \left| \int_{a_j}^{b_j} \gamma(s) ds \right| \leq \int_{a_j}^{b_j} |\gamma(s)| ds = \zeta(b_j) - \zeta(a_j) = |\zeta(b_j) - \zeta(a_j)|,$$

we deduce that

$$\sum_{j=1}^n |\eta(b_j) - \eta(a_j)| \leq \sum_{j=1}^n |\zeta(b_j) - \zeta(a_j)| < \varepsilon$$

for all intervals as before. Hence η is absolutely continuous.]

Give $C([a, b], E)$ the topology of uniform convergence (defined by the seminorms $\|\cdot\|_{\mathcal{L}^\infty, q}$ with $q \in P(E)$). Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., a bifunctor on sequentially complete (FEP)-spaces, resp., a bifunctor on integral complete locally convex spaces) and E be a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space). It is useful to note:

Lemma 3.9 *The map $\Psi: AC_{\mathcal{E}}([a, b], E) \rightarrow C([a, b], E) \times \mathcal{E}([a, b], E)$, $\eta \mapsto (\eta, \eta')$ is a linear topological embedding with closed image.*

Proof. The linearity is clear. Let $q \in P(E)$. Since $\eta(t) = \eta(t_0) + \int_{t_0}^t \eta'(s) ds$, we have

$$q(\eta(t)) \leq q(\eta(t_0)) + q\left(\int_{t_0}^t \eta'(s) ds\right) \leq q(\eta(t_0)) + \|\eta'\|_{L^1, q}$$

for all $t \in [a, b]$ and thus $\|\eta\|_{\mathcal{L}^\infty, q} \leq q(\eta(t_0)) + \|\eta'\|_{L^1, q}$. As a function of η , this is a continuous seminorm on $AC_{L^1}([a, b], E)$ (resp., on $AC_{L_{rc}^\infty}([a, b], E)$)²⁰ and hence on $AC_{\mathcal{E}}([a, b], E)$. Thus Ψ is continuous, entailing that the initial topology \mathcal{O}_Ψ on $AC_{\mathcal{E}}([a, b], E)$ with respect to Ψ is coarser than the initial topology \mathcal{O}_Φ with respect to Φ . The evaluation map $\text{ev}_{t_0}: C([a, b], E) \rightarrow E$ is continuous linear. Since $\Phi = (\text{ev}_{t_0} \times \text{id}) \circ \Psi$, we get the converse inclusion $\mathcal{O}_\Phi \subseteq \mathcal{O}_\Psi$ and hence equality, $\mathcal{O}_\Phi = \mathcal{O}_\Psi$. Hence Ψ is a topological embedding. Now consider a net $(\eta_\alpha, \eta'_\alpha)_{\alpha \in A}$ in $\text{im}(\Psi)$ such that $(\eta_\alpha \rightarrow \eta'_\alpha) \rightarrow (\eta, \gamma)$ in $C([a, b], E) \times \mathcal{E}([a, b], E)$. Define $\zeta = \Phi^{-1}(\eta(t_0), \gamma) \in AC_{\mathcal{E}}([a, b], E)$. This is the function $[a, b] \rightarrow E$,

$$t \mapsto \eta(t_0) + \int_{t_0}^t \gamma(s) ds.$$

For $q \in P(E)$, we have

$$\begin{aligned} q(\eta_\alpha(t) - \zeta(t)) &= q\left(\text{ev}_{t_0}(\eta_\alpha - \eta(t_0)) + \int_{t_0}^t (\eta'_\alpha(s) - \gamma'(s)) ds\right) \\ &\leq \|\eta_\alpha - \eta\|_{\mathcal{L}^\infty, q} + \|\eta'_\alpha - \gamma\|_{L^1, q} \end{aligned}$$

²⁰Recalling that $\|\cdot\|_{L^1, q} \leq (b-a)\|\cdot\|_{L^\infty, q}$ on $L_{rc}^\infty([a, b], E)$.

for each $t \in [a, b]$ and hence

$$\|\eta_\alpha - \zeta\|_{\mathcal{L}^\infty, q} \leq \|\eta_\alpha - \eta\|_{\mathcal{L}^\infty, q} + \|\eta'_\alpha - \gamma\|_{L^1, q} \rightarrow 0.$$

As the left hand side converges to $\|\eta - \zeta\|_{\mathcal{L}^\infty, q}$, we see that $\eta = \zeta$, whence $\eta \in AC_{\mathcal{E}}([a, b], E)$ with $\eta' = \gamma$ and thus $(\eta, \gamma) = \Psi(\eta) \in \text{im}(\Psi)$. \square

Remark 3.10 If $t_1 \in [a, b]$ and $\eta \in AC_{\mathcal{E}}([a, b], E)$ with respect to $t_0 \in [a, b]$, with $\eta' = [\gamma]$, then

$$\eta(t_1) = \eta(t_0) + \int_{t_0}^{t_1} \gamma(s) ds,$$

whence $\eta(t)$ equals

$$\eta(t_0) + \int_{t_0}^t \gamma(s) ds = \eta(t_1) + \eta(t_0) - \eta(t_1) + \int_{t_0}^t \gamma(s) ds = \eta(t_1) + \int_{t_1}^t \gamma(s) ds.$$

We deduce that η is also in $AC_{\mathcal{E}}([a, b], E)$ with respect to t_1 . Since the map $AC_{\mathcal{E}}([a, b], E) \rightarrow E$, $\eta \mapsto \eta(t_1) = \text{ev}_{t_1} \circ \text{pr}_1 \circ \Psi$ (with Ψ as in Lemma 3.10) is continuous, we deduce that the topology on $AC_{\mathcal{E}}([a, b], E)$ with respect to t_1 is coarser than that with respect to t_0 , and reversing the roles of t_0 and t_1 we deduce that both topologies are equal.

Remark 3.11 By Lemma 3.9, the inclusion map

$$j: AC_{\mathcal{E}}([a, b], E) \rightarrow C([a, b], E)$$

is continuous. If $V \subseteq E$ is an open subset, then $C([a, b], V)$ is open in $C([a, b], E)$. Thus

$$AC_{\mathcal{E}}([a, b], V) := \{\eta \in AC_{\mathcal{E}}([a, b], E) : \eta([a, b]) \subseteq V\}$$

is open in $AC_{\mathcal{E}}([a, b], E)$ (as $AC_{\mathcal{E}}[a, b], V) = j^{-1}(C([a, b], V))$).

3.12 If $a \leq \alpha < \beta \leq b$ and $[\gamma] \in \mathcal{E}([a, b], E)$, then $[\gamma|_{[\alpha, \beta]}] \in \mathcal{E}([\alpha, \beta], E)$ by axiom (B2), taking $c := \alpha$ and $d := \beta$ there. As a consequence, if $\eta \in AC_{\mathcal{E}}([a, b], E)$, then $\eta|_{[\alpha, \beta]} \in AC_{\mathcal{E}}([\alpha, \beta], E)$.

Conversely, we'd like to check the \mathcal{E} -property on subintervals.

Definition 3.13 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces). We say that \mathcal{E} satisfies the “locality axiom” if the following condition is satisfied:

- (Loc) For each Fréchet space (resp., sequentially complete (FEP)-space, resp., integral complete locally convex space) E , $n \in \mathbb{N}$ and real numbers $a = t_0 < t_1 < \cdots < t_n = b$, the map

$$\mathcal{E}([a, b], E) \rightarrow \prod_{j=1}^n \mathcal{E}([t_{j-1}, t_j], E), \quad [\gamma] \mapsto ([\gamma|_{[t_{j-1}, t_j]}])_{j=1, \dots, n}$$

is an isomorphism of topological vector spaces.

Remark 3.14 The locality axiom immediately implies the following useful fact: If $\gamma: [a, b] \rightarrow E$ is a measurable map and $a = t_0 < t_1 < \cdots < t_n = b$, then $[\gamma] \in \mathcal{E}([a, b], E)$ if and only if $[\gamma|_{[t_{j-1}, t_j]}] \in \mathcal{E}([t_{j-1}, t_j], E)$ for all $j \in \{1, \dots, n\}$.

3.15 If \mathcal{E} is any of L^p ($p \in [1, \infty]$), L_{rc}^∞ or R , then \mathcal{E} satisfies the locality axiom (for trivial reasons).

3.16 The locality axiom implies that if $\eta: [a, b] \rightarrow E$ is continuous, $a = t_0 < t_1 < \cdots < t_n = b$ and $\eta|_{[t_{j-1}, t_j]} \in AC_{\mathcal{E}}([t_{j-1}, t_j], E)$ for all $j \in \{1, \dots, n\}$, then $\eta \in AC_{\mathcal{E}}([a, b], E)$.

Definition 3.17 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) and $k \in \mathbb{N} \cup \{\infty\}$. We say that C^k -functions act on $AC_{\mathcal{E}}$ if the following condition is satisfied:

- (A_k) For all Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) E and F , each C^k -map $f: V \rightarrow F$ on an open subset $V \subseteq E$, all $a < b$ in \mathbb{R} and each $\eta \in AC_{\mathcal{E}}([a, b], E)$ such that $\eta([a, b]) \subseteq V$, we have

$$f \circ \eta \in AC_{\mathcal{E}}([a, b], F).$$

Lemma 3.18 (a) Let \mathcal{E} be L^p for $p \in [1, \infty]$, L_{rc}^∞ or R , considered as bifunctors on Fréchet spaces (or on sequentially complete (FEP)-spaces). Then C^1 -functions act on $A_{\mathcal{E}}$.

(b) C^2 -functions act on $AC_{L_{rc}^\infty}$ and on AC_R , if L_{rc}^∞ and R are considered as bifunctors on integral complete locally convex spaces.

Proof. (a) Let E and F be Fréchet spaces (or sequentially complete (FEP)-spaces), $a, b \in \mathbb{R}$ with $a < b$ and $\eta \in AC_{\mathcal{E}}([a, b], E)$. Write $\eta' = [\gamma]$. Let $V \subseteq E$ be an open subset such that $\eta([a, b]) \subseteq V$ and $f: V \rightarrow F$ be a C^1 -map.

Step 1. The map $f \circ \eta: [a, b] \rightarrow F$ is continuous. Also the map

$$df: V \times E \rightarrow F$$

is continuous, and thus $\theta := df \circ (\eta, \gamma) \in \mathcal{E}([a, b], F)$, by Lemma 2.1. We can therefore form a map

$$\zeta: [a, b] \rightarrow F, \quad \zeta(t) := f(\eta(a)) + \int_a^t df(\eta(s), \gamma(s)) ds.$$

Then $\zeta \in AC_{\mathcal{E}}([a, b], F)$, $\zeta'(t) = [\theta]$ and $\zeta(a) = f(\eta(a))$. If we can show that $\lambda \circ \zeta = \lambda \circ f \circ \eta$ for each $\lambda \in F'$, then $f \circ \eta = \zeta \in AC_{\mathcal{E}}([a, b], F)$ (by the Hahn-Banach Separation Theorem). We may therefore assume now that $F = \mathbb{R}$.

Step 2. We claim that $f \circ \eta \in AC_{L^1}([a, b], \mathbb{R})$. If this is true, then

$$(f \circ \eta)'(t) = df(\eta(t), \gamma(t)) = \zeta'(t)$$

for almost all $t \in [a, b]$ (by Lemmas 1.28 and 1.57) and hence $f \circ \eta = \zeta$ since both $f \circ \eta$ and ζ are absolutely continuous, $(f \circ \eta)' = \zeta'$ and $f(\eta(a)) = \zeta(a)$.

To prove the claim, we use Lemma 1.60 to find a continuous seminorm $q \in P(E)$ and an open set $V_1 \subseteq V$ with $\eta([a, b]) \subseteq V_1$ such that

$$|f(z) - f(y)| \leq q(z - y) \quad \text{for all } z, y \in V_1.$$

We have $q \circ \gamma \in \mathcal{L}^1([a, b], \mathbb{R})$, whence

$$\sigma: [a, b] \rightarrow \mathbb{R}, \quad \sigma(t) := \int_a^t q(\gamma(s)) ds$$

is monotonically increasing and absolutely continuous. If $\varepsilon > 0$, let $\delta \in]0, \rho]$ such that

$$\sum_{j=1}^n |\sigma(b_j) - \sigma(a_j)| < \varepsilon$$

for each $n \in \mathbb{N}$ and disjoint intervals $]a_1, b_1[, \dots,]a_n, b_n[$ in $[a, b]$ of total length $\sum_{j=1}^n (b_j - a_j) < \delta$ (see [60, Theorem 7.18]). For each $j \in \{1, \dots, n\}$, we have

$$\eta(b_j) - \eta(a_j) = \int_{a_j}^{b_j} \gamma(t) dt$$

and hence

$$q(\eta(b_j) - \eta(a_j)) \leq \int_{a_j}^{b_j} q(\gamma(t)) dt = \sigma(b_j) - \sigma(a_j) = |\sigma(b_j) - \sigma(a_j)|.$$

Thus

$$\sum_{j=1}^n |f(\eta(b_j)) - f(\eta(a_j))| \leq \sum_{j=1}^n q(\eta(b_j) - \eta(a_j)) \leq \sum_{j=1}^n |\sigma(b_j) - \sigma(a_j)| < \varepsilon.$$

Hence $f \circ \eta$ is absolutely continuous (by [60, Theorem 7.18]), as required.

(b) Let E and F be integral complete locally convex spaces, $a, b \in \mathbb{R}$ with $a < b$ and $\eta \in AC_{\mathcal{E}}([a, b], E)$. Write $\eta' = [\gamma]$. Let $V \subseteq E$ be an open subset such that $\eta([a, b]) \subseteq V$ and $f: V \rightarrow F$ be a C^2 -map. Step 1 of the proof of (a) applies without changes. We may therefore assume now that $F = \mathbb{R}$. By Lemma 1.63, we find an open subset $V_1 \subseteq V$ with $\eta([a, b]) \subseteq V_1$, a continuous linear map $\lambda: E \rightarrow Y$ to a Banach space Y and a C^1 -function $g: W \rightarrow \mathbb{R}$ on an open subset $W \subseteq Y$ with $\lambda(V_1) \subseteq W$ such that

$$f|_{V_1} = g \circ \lambda|_{V_1}.$$

Now $\lambda \circ \eta \in AC_{\mathcal{E}}([a, b], Y)$ since

$$\lambda(\eta(t)) = \lambda \left(\int_a^t \gamma(s) ds \right) = \int_a^t \lambda(\gamma(s)) ds$$

for each $t \in [a, b]$ (by 1.17), where $[\lambda \circ \gamma] = \mathcal{E}([a, b], \lambda)([\gamma]) \in \mathcal{E}([a, b], Y)$ by axiom (B1). Thus $f \circ \eta = g \circ (\lambda \circ \eta) \in AC_{\mathcal{F}}([a, b], \mathbb{R}) = AC_{\mathcal{E}}([a, b], \mathbb{R})$ by (a), if we set $\mathcal{F}([a, b], H) := \mathcal{E}([a, b], H)$ for each Fréchet space H . \square

Remark 3.19 The preceding proof shows that

$$(f \circ \eta)' = [f \circ (\eta, \gamma)]$$

if $f: E \supseteq V \rightarrow F$ is C^1 (resp., C^2), $\eta \in AC_{\mathcal{E}}([a, b], V)$ and $\eta' = [\gamma]$.

Definition 3.20 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) such that C^k -functions act on \mathcal{E} for some $k \in \mathbb{N}$ and the locality axiom (Loc) from Definition 3.13 is satisfied. Let M be a C^k -manifold modelled on a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space) E . If $a, b \in \mathbb{R}$ with $a < b$, we let $AC_{\mathcal{E}}([a, b], M)$ be the set of all functions

$$\eta: [a, b] \rightarrow M$$

such that η is continuous and there is a partition $a = t_0 < t_1 < \dots < t_n = b$ of $[a, b]$ such that, for each $j \in \{1, \dots, n\}$, there exists a chart $\phi_j: U_j \rightarrow V_j \subseteq E$ of M such that $\eta([t_{j-1}, t_j]) \subseteq U_j$ and $\phi_j \circ \eta|_{[t_{j-1}, t_j]} \in AC_{\mathcal{E}}([t_{j-1}, t_j], E)$.

We imposed the axioms (A_k) and (Loc) to ensure the independence of the $AC_{\mathcal{E}}$ -property from the chosen partition:

Lemma 3.21 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) such that C^k -functions act on $A_{\mathcal{E}}$ for some $k \in \mathbb{N} \cup \{\infty\}$ and \mathcal{E} satisfies the locality axiom (Loc) from Definition 3.13. Let E be a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space). Let $a, b \in \mathbb{R}$ such that $a < b$ and M be a C^k -manifold modelled on E . If $\eta \in AC_{\mathcal{E}}([a, b], M)$, then*

$$\phi \circ \eta|_{[\alpha, \beta]} \in AC_{\mathcal{E}}([\alpha, \beta], M)$$

for each chart $\phi: U \rightarrow V \subseteq E$ of M and all $\alpha < \beta$ in J such that $\eta([\alpha, \beta]) \subseteq U$. In particular, $AC_{\mathcal{E}}([a, b], E)$ for E as a vector space coincides as a set with $AC_{\mathcal{E}}([a, b], E)$ for E considered as a manifold modelled on E with the maximal C^k -atlas associated with the global chart id_E .

Proof. Let $a = t_0 < t_1 < \dots < t_n = b$ and charts $\phi_j: U_j \rightarrow V_j \subseteq E$ be as in Definition 3.20. Thus

$$\phi_j \circ \eta|_{[t_{j-1}, t_j]} \in AC_{\mathcal{E}}([t_{j-1}, t_j], E) \quad \text{for all } j \in \{1, \dots, n\}.$$

Using ϕ_j as the chart for the new points, we may add additional points inside $[t_{j-1}, t_j]$. We may therefore assume that $\alpha = t_k$ and $\beta = t_\ell$ for certain $k, \ell \in \{1, \dots, n\}$ with $k < \ell$. Then $\eta([t_{j-1}, t_j]) \subseteq U_j \cap U$ for all $j \in \{k+1, \dots, \ell\}$ and the transition map

$$\tau_j := \phi \circ \phi_j^{-1}: \phi_j(U_j \cap U) \rightarrow \phi(U_j \cap U)$$

is a C^1 -diffeomorphism between open subsets of E . We have to show that $\phi \circ \eta|_{[\alpha, \beta]} \in AC_{\mathcal{E}}([\alpha, \beta], E)$. By the locality axiom, it suffices to show that $\phi \circ \eta|_{[t_{j-1}, t_j]} \in AC_{\mathcal{E}}([t_{j-1}, t_j], E)$ for $j \in \{k+1, \dots, \ell\}$. But this is true, since $\phi \circ \eta|_{[\alpha, \beta]} = \tau_j \circ (\phi_j \circ \eta|_{[t_{j-1}, t_j]})$ and C^k -functions (like the τ_j) act on $AC_{\mathcal{E}}$. \square

Remark 3.22 Let $\eta: [a, b] \rightarrow M$ be a continuous function such that, for each $s \in [a, b]$, there is $\varepsilon_s > 0$ such that $\eta([a, b] \cap [s - \varepsilon_s, s + \varepsilon_s]) \subseteq U_s$ for some chart $\phi_s: U_s \rightarrow V_s \subseteq E$ of M and

$$\phi_s \circ \eta|_{[a, b] \cap [s - \varepsilon_s, s + \varepsilon_s]} \in AC_{\mathcal{E}}([a, b] \cap [s - \varepsilon_s, s + \varepsilon_s], E).$$

Then $\eta \in AC_{\mathcal{E}}([a, b], M)$.

In fact, $([a, b] \cap [s - \varepsilon_s, s + \varepsilon_s])_{s \in [a, b]}$ is an open cover of the compact metric space $[a, b]$. Lebesgue's Lemma provides a Lebesgue number $\delta > 0$ for this open cover. Thus, for each $t \in [a, b]$, there exists $s(t) \in [a, b]$ such that $[a, b] \cap [t - \delta, t + \delta] \subseteq [a, b] \cap [s(t) - \varepsilon_{s(t)}, s(t) + \varepsilon_{s(t)}]$. Choose $a = t_0 < t_1 < \dots < t_n = b$ such that $t_j - t_{j-1} < \delta$ for all $j \in \{1, \dots, n\}$. Then $[t_{j-1}, t_{j+1}] \subseteq [a, b] \cap [s(t_{j-1}) - \varepsilon_{s(t_{j-1})}, s(t_{j-1}) + \varepsilon_{s(t_{j-1})}]$ and hence $\phi_{s(t_{j-1})} \circ \eta|_{[t_{j-1}, t_{j+1}]} \in AC_{\mathcal{E}}([t_{j-1}, t_{j+1}], E)$ for all $j \in \{1, \dots, n\}$. Thus $\eta \in AC_{\mathcal{E}}([a, b], E)$.

Definition 3.23 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) which satisfies the locality axiom (Loc). Let E be a Fréchet space (resp., a sequentially complete (FEP)-spaces, resp., an integral complete locally convex space) and $I \subseteq \mathbb{R}$ be a non-degenerate interval. We let $AC_{\mathcal{E}}(I, E)$ be the vector space of all continuous mappings $\eta: I \rightarrow E$ such that $\eta|_{[a, b]} \in AC_{\mathcal{E}}([a, b], E)$ for all compact intervals $[a, b] \subseteq I$. If smooth maps act on $AC_{\mathcal{E}}$ and M is a smooth manifold modelled on E , we let $AC_{\mathcal{E}}(I, M)$ be the set of all continuous mappings $\eta: I \rightarrow M$ such that $\eta|_{[a, b]} \in AC_{\mathcal{E}}([a, b], M)$ for all compact intervals $[a, b] \subseteq I$.

Lemma 3.24 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) such that C^k -functions act on $A_{\mathcal{E}}$ for some $k \in \mathbb{N} \cup \{\infty\}$ and \mathcal{E} satisfies the locality axiom (Loc) from Definition 3.13. Let E and F be Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces), $a, b \in \mathbb{R}$ such that $a < b$ and*

$$f: M \rightarrow N$$

be a C^k -map, where M be a C^k -manifold modelled on E and N be a C^k -manifold modelled on F . Then $f \circ \eta \in AC_{\mathcal{E}}([a, b], N)$ for all $\eta \in AC_{\mathcal{E}}([a, b], M)$, enabling us to define a map

$$AC_{\mathcal{E}}([a, b], f): AC_{\mathcal{E}}([a, b], M) \rightarrow AC_{\mathcal{E}}([a, b], N), \quad \eta \mapsto f \circ \eta.$$

Proof. For each $t \in [a, b]$, there is a chart $\psi_t: P_t \rightarrow Q_t \subseteq F$ of N such that $f(\eta(t)) \in P_t$ and a chart $\phi_t: U_t \rightarrow V_t \subseteq E$ of M such that $\eta(t) \in U_t$. After shrinking U_t , we may assume that $f(U_t) \subseteq P_t$. There is $\varepsilon > 0$ such that $[a, b] \cap [t - \varepsilon, t + \varepsilon] \subseteq \eta^{-1}(U_t)$. Then $\phi_t \circ \eta|_{[a, b] \cap [t - \varepsilon, t + \varepsilon]} \in AC_{\mathcal{E}}([a, b] \cap [t - \varepsilon, t + \varepsilon], E)$ (by Lemma 3.21). Since

$$\psi_t \circ f \circ \eta|_{[a, b] \cap [t - \varepsilon, t + \varepsilon]} = (\psi_t \circ f \circ (\phi_t)^{-1})|_{V_t} \circ (\phi_t \circ \eta|_{[a, b] \cap [t - \varepsilon, t + \varepsilon]})$$

and C^k -functions (like $(\psi_t \circ f \circ (\phi_t)^{-1})|_{V_t}$) act on $AC_{\mathcal{E}}$, we see that $\psi_t \circ f \circ \eta|_{[a, b] \cap [t - \varepsilon, t + \varepsilon]} \in AC_{\mathcal{E}}([a, b] \cap [t - \varepsilon, t + \varepsilon], F)$. Hence $f \circ \eta \in AC_{\mathcal{E}}([a, b], N)$, by Remark 3.22. \square

Definition 3.25 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces). We say that *smooth functions act smoothly on $AC_{\mathcal{E}}$* if C^{∞} -functions act on $AC_{\mathcal{E}}$ and the following holds:

(S) The map

$$AC_{\mathcal{E}}([a, b], f): AC_{\mathcal{E}}([a, b], V) \rightarrow AC_{\mathcal{E}}([a, b], F), \quad \eta \mapsto f \circ \eta$$

is C^{∞} , for all $a < b$ in \mathbb{R} , Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) E and F , each open subset $V \subseteq E$ and each smooth map $f: V \rightarrow F$.

Lemma 3.26 *In the situation of Definition 3.25, the map*

$$h := AC_{\mathcal{E}}([a, b], f): AC_{\mathcal{E}}([a, b], V) \rightarrow AC_{\mathcal{E}}([a, b], F)$$

satisfies

$$dh = AC_{\mathcal{E}}([a, b], df), \quad (18)$$

identifying $AC_{\mathcal{E}}([a, b], E)^2$ with $AC_{\mathcal{E}}([a, b], E \times E)$.

Proof. It is well known that the map

$$g := C([a, b], f): C([a, b], V) \rightarrow C([a, b], F), \quad \eta \mapsto f \circ \eta$$

is C^∞ and

$$dg = C([a, b], df): C([a, b], V \times E) \rightarrow C([a, b], F), \quad (\eta_1, \eta_2) \mapsto df \circ (\eta_1, \eta_2)$$

if we identify $C([a, b], V) \times C([a, b], E)$ with $C([a, b], V \times E)$ (cf. [20] or [38]). As we assume that smooth functions act smoothly on $AC_{\mathcal{E}}$, we know that h is smooth. Let $j_E: AC_{\mathcal{E}}([a, b], E) \rightarrow C([a, b], E)$ and $j_F: AC_{\mathcal{E}}([a, b], F) \rightarrow C([a, b], F)$ be the inclusion maps, which are continuous linear (see Remark 3.11). Then $j_F \circ h = g \circ j_E|_{AC_{\mathcal{E}}([a, b], V)}$ and the Chain Rule yields:

$$j_F \circ dh = dg \circ (j_E|_{AC_{\mathcal{E}}([a, b], V)} \times j_E).$$

Thus $dh(\eta_1, \eta_2) = j_F(dh(\eta_1, \eta_2)) = dg(\eta_1, \eta_2) = df \circ (\eta_1, \eta_2)$ for all $\eta_1 \in AC_{\mathcal{E}}([a, b], V)$ and $\eta_2 \in AC_{\mathcal{E}}([a, b], E)$. Thus (18) is valid and the proof is complete. \square

Lemma 3.27 *Smooth functions act smoothly on AC_{L^p} for L^p as a bifunctor on Fréchet spaces (or sequentially complete (FEP)-spaces), for each $p \in [1, \infty]$. Moreover, smooth functions act smoothly on $AC_{L_{rc}^\infty}$ and AC_R , for L_{rc}^∞ and R considered as bifunctors on integral complete locally convex spaces.*

This follows from the following more detailed result:

Lemma 3.28 *Let \mathcal{E} be L^p for some $p \in [1, \infty]$, and E as well as F be sequentially complete (FEP)-spaces (e.g., Fréchet spaces). Or let \mathcal{E} be L_{rc}^∞ or R , and let E as well as F be integral complete locally convex spaces. Let*

$a < b$ be real numbers, $V \subseteq E$ be open and $f: V \rightarrow F$ be a C^{k+2} -map for some $k \in \mathbb{N}_0 \cup \{\infty\}$. Then the map

$$AC_{\mathcal{E}}([a, b], f): AC_{\mathcal{E}}([a, b], V) \rightarrow AC_{\mathcal{E}}([a, b], F), \quad \eta \mapsto f \circ \eta$$

is C^k . If $k \geq 1$, then

$$dg = AC_{\mathcal{E}}([a, b], df), \quad (19)$$

identifying $AC_{\mathcal{E}}([a, b], E)^2$ with $AC_{\mathcal{E}}([a, b], E \times E)$.

Proof. Note first that $[f \circ \eta] \in AC_{\mathcal{E}}([a, b], F)$ by Lemma 3.18, since f is at least C^2 . Let $j_E: AC_{\mathcal{E}}([a, b], E) \rightarrow C([a, b], E)$ and $j_F: AC_{\mathcal{E}}([a, b], F) \rightarrow C([a, b], F)$ be the inclusion maps, which are continuous linear (see Remark 3.11). Because $\Phi: AC_{\mathcal{E}}([a, b], F) \rightarrow C([a, b], F) \times \mathcal{E}([a, b], F)$ is a topological embedding with closed image, [5, Lemmas 10.1 and 10.2] show that g will be C^k if we can prove that

$$j_F \circ AC_{\mathcal{E}}([a, b], f) = C([a, b], f) \circ j_E|_{AC_{\mathcal{E}}([a, b], V)} \quad (20)$$

and $D_F \circ AC_{\mathcal{E}}([a, b], f)$ are C^k , where

$$D_F: AC_{\mathcal{E}}([a, b], F) \rightarrow \mathcal{E}([a, b], F), \quad \eta \mapsto \eta'$$

is the differentiation operator (which is continuous linear). Since

$$C([a, b], f): C([a, b], V) \rightarrow C([a, b], F), \quad \eta \mapsto f \circ \eta$$

is C^{k+2} and hence C^k (see [38], cf. [20]), we deduce from (20) that $j \circ AC_{\mathcal{E}}([a, b], f)$ is C^k . Let $\eta' = [\gamma]$. Then $D_F(f \circ \eta) = [df \circ (\eta, \gamma)]$ by Remark 3.19, and thus

$$D_F \circ AC_{\mathcal{E}}([a, b], f) = \widetilde{df}$$

with $\widetilde{f}: C([a, b], V) \times \mathcal{E}([a, b], E) \rightarrow \mathcal{E}([a, b], F)$, $(\eta, [\gamma]) \mapsto [df \circ (\eta, \gamma)]$. Since df is C^{k+1} , Proposition 2.3 (resp., Proposition 2.2) show that \widetilde{f} is C^k .

To verify the validity of (19), abbreviate $h := C([a, b], f)$. Then $j_F \circ g = h \circ j_E|_{AC_{\mathcal{E}}([a, b], V)}$ (by (20)) and the Chain Rule yields:

$$j_F \circ dg = dh \circ (j_E|_{AC_{\mathcal{E}}([a, b], V)} \times j_E).$$

Since $dh = C([a, b], df)$ (cf. [20]), we see that $dg(\eta_1, \eta_2) = dh(\eta_1, \eta_2) = df \circ (\eta_1, \eta_2)$ for all $\eta_1 \in AC_{\mathcal{E}}([a, b], V)$ and $\eta_2 \in AC_{\mathcal{E}}([a, b], E)$. Thus (19) is valid and the proof is complete. \square

Lemma 3.29 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) which satisfies the locality axiom and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let E and F be Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $V \subseteq E$ be an open set and $f: V \rightarrow F$ be a \mathbb{K} -analytic map. Then also the map*

$$AC_{\mathcal{E}}([a, b], f): AC_{\mathcal{E}}([a, b], V) \rightarrow AC_{\mathcal{E}}([a, b], F), \quad \eta \mapsto f \circ \eta$$

is \mathbb{K} -analytic, for all $a < b$ in \mathbb{R} .

Proof. Assume $\mathbb{K} = \mathbb{C}$ first. As we assume that smooth functions act smoothly on $AC_{\mathcal{E}}$, we know that $h := AC_{\mathcal{E}}([a, b], f)$ is smooth. By Lemma 3.26, we have

$$dh = AC_{\mathcal{E}}([a, b], df), \quad (\eta_1, \eta_2) \mapsto df \circ (\eta_1, \eta_2)$$

if we identify $\mathcal{E}([a, b], E \times E)$ with $\mathcal{E}([a, b], E)^2$. Since $df(x, \cdot): E \rightarrow F$ is complex linear for each x , it follows that $dh(\eta_1, \eta_2)$ is complex linear in η_2 , for each $\eta_1 \in AC_{\mathcal{E}}([a, b], E)$. This implies that the smooth map h is complex analytic (see [19] of [38]).

If $\mathbb{K} = \mathbb{R}$, let $\tilde{f}: \tilde{V} \rightarrow F_{\mathbb{C}}$ be a complex analytic extension of f to an open subset $\tilde{V} \subseteq E_{\mathbb{C}}$ which contains V . Note that both $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$ are Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces). Hence $AC_{\mathcal{E}}([a, b], \tilde{f}): AC_{\mathcal{E}}([a, b], \tilde{V}) \rightarrow AC_{\mathcal{E}}([a, b], F_{\mathbb{C}})$ is complex analytic. Here $AC_{\mathcal{E}}([a, b], F_{\mathbb{C}}) = AC_{\mathcal{E}}([a, b], F)_{\mathbb{C}}$ (cf. Remark 3.5) and $AC_{\mathcal{E}}([a, b], E_{\mathbb{C}}) = AC_{\mathcal{E}}([a, b], E)_{\mathbb{C}}$. Thus $AC_{\mathcal{E}}([a, b], \tilde{f})$ is a complex analytic extension of $AC_{\mathcal{E}}([a, b], f)$ and hence $AC_{\mathcal{E}}([a, b], f)$ is real analytic. \square

Lemma 3.30 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) which satisfies the locality axiom (Loc), and such that smooth functions act smoothly on $A_{\mathcal{E}}$. Let E_1 , E_2 and F be Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $\mathbb{K} = \mathbb{R}$, let $r \in \{\infty, \omega\}$; if $\mathbb{K} = \mathbb{C}$, let $r = \omega$. Let $a < b$ be real numbers, M be a $C_{\mathbb{K}}^r$ -manifold modelled on E_1 , $V \subseteq E_2$ be open and $f: M \times V \rightarrow F$ be a $C_{\mathbb{K}}^r$ -map. Also, let $\zeta \in AC_{\mathcal{E}}([a, b], M)$. Then the map*

$$AC_{\mathcal{E}}([a, b], V) \rightarrow AC_{\mathcal{E}}([a, b], F), \quad \eta \mapsto f \circ (\zeta, \eta)$$

is $C_{\mathbb{K}}^r$.

Proof. We find $a = t_0 < t_1 < \dots < t_n = b$ such that $\zeta([t_{j-1}, t_j]) \subseteq U_j$ for some chart $\phi_j: U_j \rightarrow V_j \subseteq E_1$ of M , for all $j \in \{1, \dots, n\}$. Then

$$f_j := f \circ ((\phi_j)^{-1} \times \text{id}_{E_2}): V_j \times E_2 \rightarrow F$$

is a $C_{\mathbb{K}}^r$ -map for $j \in \{1, \dots, n\}$. Moreover, $\zeta_j := \phi_j \circ \zeta|_{[t_{j-1}, t_j]} \in AC_{\mathcal{E}}([t_{j-1}, t_j], V_j)$ and

$$(f \circ (\zeta, \eta))|_{[t_{j-1}, t_j]} = f_j \circ (\zeta_j, \eta|_{[t_{j-1}, t_j]}). \quad (21)$$

Since \mathcal{E} satisfies the locality axiom, the map

$$AC_{\mathcal{E}}([a, b], F) \rightarrow \prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], F), \quad \eta \mapsto (\eta|_{[t_{j-1}, t_j]})_{j \in \{1, \dots, n\}}$$

is a linear topological embedding with closed image.²¹ Hence, by Lemmas 10.1 and 10.2 in [5] (and analogous lemmas for analytic maps in [38]), we need only show that the maps

$$h_j: AC_{\mathcal{E}}([a, b], V) \rightarrow AC_{\mathcal{E}}([a, b], F), \quad \eta \mapsto (f \circ (\zeta, \eta))|_{[t_{j-1}, t_j]}$$

are $C_{\mathbb{K}}^r$ for all $j \in \{1, \dots, n\}$. The mappings

$$AC_{\mathcal{E}}([a, b], E_2) \rightarrow AC_{\mathcal{E}}([t_{j-1}, t_j], E_2), \quad \eta \mapsto \eta|_{[t_{j-1}, t_j]}$$

are continuous \mathbb{K} -linear for $j \in \{1, \dots, n\}$, whence the maps

$$\rho_j: AC_{\mathcal{E}}([a, b], V) \rightarrow AC_{\mathcal{E}}([t_{j-1}, t_j], V), \quad \eta \mapsto \eta|_{[t_{j-1}, t_j]}$$

are $C_{\mathbb{K}}^r$. The map

$$AC_{\mathcal{E}}([t_{j-1}, t_j], f_j): AC_{\mathcal{E}}([t_{j-1}, t_j], V_j \times V) \rightarrow AC_{\mathcal{E}}([t_{j-1}, t_j], F)$$

is $C_{\mathbb{K}}^r$ as we assume that smooth functions act smoothly on $AC_{\mathcal{E}}$ (see Lemma 3.29 if $r = \omega$). Identifying $AC_{\mathcal{E}}([t_{j-1}, t_j], V_j \times V)$ with

$$AC_{\mathcal{E}}([t_{j-1}, t_j], V_j) \times AC_{\mathcal{E}}([t_{j-1}, t_j], V),$$

we have

$$h_j(\eta) = AC_{\mathcal{E}}([t_{j-1}, t_j], f_j)(\zeta_j, \rho_j(\eta))$$

for all $\eta \in AC_{\mathcal{E}}([a, b], V)$ (exploiting (21)). Hence h_j is $C_{\mathbb{K}}^r$, which completes the proof. \square

²¹The image consists of all $(\eta)_{j \in \{1, \dots, n\}} \in \prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], F)$ such that $\eta_j(t_j) = \eta_{j+1}(t_j)$ for all $j \in \{1, \dots, n-1\}$.

4 The Lie groups $AC_{\mathcal{E}}([0, 1], G)$

In this section, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $\mathbb{K} = \mathbb{R}$, we let $r \in \{\infty, \omega\}$; if $\mathbb{K} = \mathbb{C}$, we let $r := \omega$. We recall the local description of Lie group structures.

4.1 Let U be a group and $U \subseteq G$ be a subset such that $U = U^{-1}$ and $e \in U$. Assume that U is endowed with a $C_{\mathbb{K}}^r$ -manifold structure modelled on a locally convex topological \mathbb{K} -vector space E such that

$$U \rightarrow U, \quad x \mapsto x^{-1}$$

is $C_{\mathbb{K}}^r$, $D_U := \{(x, y) \in U \times U : xy \in U\}$ is open in $U \times U$ and the map

$$U \times U \rightarrow U, \quad (x, y) \mapsto xy$$

is $C_{\mathbb{K}}^r$. Also, assume that for each $g \in G$, there is an open identity neighbourhood $W \subseteq U$ such that $gWg^{-1} \subseteq U$, and that the map

$$W \rightarrow U, \quad x \mapsto gxg^{-1}$$

is $C_{\mathbb{K}}^r$. Then there is a unique $C_{\mathbb{K}}^r$ -manifold structure on G modelled on E which makes G a $C_{\mathbb{K}}^r$ -Lie group and such that G has U (with its original manifold structure) as an open $C_{\mathbb{K}}^r$ -submanifold.

Proposition 4.2 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) which satisfies the locality axiom (Loc) and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let G be a $C_{\mathbb{K}}^r$ -Lie group modelled on a Fréchet space (resp., on a sequentially complete (FEP)-space, resp., on an integral complete locally convex space) E over \mathbb{K} . Then $AC_{\mathcal{E}}([0, 1], G)$ is a group under pointwise multiplication and there is a unique $C_{\mathbb{K}}^r$ -Lie group structure on $AC_{\mathcal{E}}([0, 1], G)$ such that*

$$AC_{\mathcal{E}}([0, 1], U) := \{\eta \in AC_{\mathcal{E}}([0, 1], G) : \eta([0, 1]) \subseteq U\}$$

is open in $AC_{\mathcal{E}}([0, 1], G)$ and

$$AC_{\mathcal{E}}([0, 1], \phi) : AC_{\mathcal{E}}([0, 1], U) \rightarrow AC_{\mathcal{E}}([0, 1], V)$$

is a $C_{\mathbb{K}}^r$ -diffeomorphism for each chart $\phi : U \rightarrow V$ of G such that $e \in U$ and $U = U^{-1}$.

Proof. *Existence of a Lie group structure.* Let $m_G: G \times G \rightarrow G$ be the group multiplication and $j_G: G \rightarrow G$ be the inversion map, $x \mapsto x^{-1}$. Using Lemma 3.24, we see that $\tilde{G} := AC_{\mathcal{E}}([0, 1], G)$ becomes a group if we define the multiplication $m_{\tilde{G}}$ via

$$\eta_1 \eta_2 := AC_{\mathcal{E}}([0, 1], m_G)(\eta_1, \eta_2)$$

(identifying $AC_{\mathcal{E}}([0, 1], G) \times AC_{\mathcal{E}}([0, 1], G)$ with $AC_{\mathcal{E}}([0, 1], G \times G)$) and the inversion $j_{\tilde{G}}$ via

$$\eta^{-1} := AC_{\mathcal{E}}([0, 1], j_G)(\eta)$$

for $\eta, \eta_1, \eta_2 \in AC_{\mathcal{E}}([0, 1], G)$. Pick a chart $\phi: U \rightarrow V$ of G such that $e \in U$ and $U = U^{-1}$. Then $\tilde{V} := AC_{\mathcal{E}}([0, 1], V)$ is an open subset of $AC_{\mathcal{E}}([0, 1], E)$. By Lemma 3.21, the map

$$\tilde{\phi} := AC_{\mathcal{E}}([0, 1], \phi): AC_{\mathcal{E}}([0, 1], U) \rightarrow AC_{\mathcal{E}}([0, 1], V), \quad \eta \mapsto \phi \circ \eta$$

is a bijection. We endow $\tilde{U} := AC_{\mathcal{E}}([0, 1], U)$ with the $C_{\mathbb{K}}^r$ -manifold structure which turns $AC_{\mathcal{E}}([0, 1], \phi)$ into a global chart. Then

$$D_U := \{(x, y) \in U \times U : xy \in U\}$$

is open in $U \times U$, and hence $D_V := (\phi \times \phi)(D_U)$ is open in $V \times V$. The mappings

$$j: V \rightarrow V, \quad j(x) := \phi((\phi^{-1}(x))^{-1})$$

and

$$m: D_V \rightarrow V, \quad (x, y) \mapsto \phi(\phi^{-1}(x)\phi^{-1}(y))$$

(which express the group inversion and group multiplication in the local chart) are $C_{\mathbb{K}}^r$. Now $AC_{\mathcal{E}}([0, 1], V)$ is an open subset of $AC_{\mathcal{E}}([0, 1], E)$ and $AC_{\mathcal{E}}([0, 1], D_V)$ is an open subset of $AC_{\mathcal{E}}([0, 1], E \times E) \cong AC_{\mathcal{E}}([0, 1], E) \times AC_{\mathcal{E}}([0, 1], E)$. Identifying $AC_{\mathcal{E}}([0, 1], G \times G)$ with the direct product $AC_{\mathcal{E}}([0, 1], G) \times AC_{\mathcal{E}}([0, 1], G)$ as a set, we have

$$\begin{aligned} AC_{\mathcal{E}}([0, 1], D_U) &= \{(\eta_1, \eta_2) \in AC_{\mathcal{E}}([0, 1], G \times G) : (\forall t \in [0, 1]) (\eta_1(t), \eta_2(t)) \in D_U\} \\ &= \{(\eta_1, \eta_2) \in \tilde{U} \times \tilde{U} : \eta_1 \eta_2 \in \tilde{U}\} =: D_{\tilde{U}}. \end{aligned}$$

Since smooth functions act smoothly on $AC_{\mathcal{E}}$ (and appealing to Lemma 3.29 if $r = \omega$), the map

$$j_{\tilde{G}} = (\tilde{\phi})^{-1} \circ AC_{\mathcal{E}}([0, 1], j) \circ \tilde{\phi}$$

is $C_{\mathbb{K}}^r$. Likewise,

$$m_{\tilde{G}} = (\tilde{\phi})^{-1} \circ AC_{\mathcal{E}}([0, 1], m) \circ (\tilde{\phi} \times \tilde{\phi})|_{D_{\tilde{U}}}$$

is $C_{\mathbb{K}}^r$. If $\zeta \in \tilde{G}$, then $K := \zeta([0, 1]) \subseteq G$ is compact. Because the map

$$h: G \times G \rightarrow G, \quad h(x, y) := xyx^{-1}$$

is continuous and $h(K \times \{e\}) = \{e\} \subseteq U$, the Wallace Lemma (see 1.1) provides open sets $P, W \subseteq G$ such that $K \times \{e\} \subseteq P \times W \subseteq h^{-1}(U)$. We may assume that $P \subseteq U$. As a consequence,

$$\zeta\eta\zeta^{-1} = h \circ (\zeta, \eta) \in AC_{\mathcal{E}}([0, 1], U) = \tilde{U}$$

for all $\eta \in AC_{\mathcal{E}}([0, 1], W) := \widetilde{W}$. The map

$$f: P \times W \rightarrow V, \quad f(x, y) := \phi(h(x, \phi^{-1}(y)))$$

is smooth, and the map $\widetilde{W} \rightarrow \tilde{U}$,

$$\eta \mapsto \zeta\eta\zeta^{-1} = h \circ (\zeta, \eta) = \tilde{\phi}^{-1}(f \circ (\zeta, \tilde{\phi}(\eta)))$$

is $C_{\mathbb{K}}^r$ by Lemma 3.30. Hence 4.1 provides a unique $C_{\mathbb{K}}^r$ -Lie group structure on \tilde{G} modelled on $AC_{\mathcal{E}}([a, b], E)$ such that \tilde{U} is an open submanifold.

Uniqueness of the Lie group structure. If also $\phi_1: U_1 \rightarrow V_1$ is a chart of G such that $e \in U_1$ and $U_1 = (U_1)^{-1}$, then $U \cap U_1$ is open in U and hence $AC_{\mathcal{E}}([0, 1], \phi(U \cap U_1))$ is open in \tilde{U} , entailing that $AC_{\mathcal{E}}([0, 1], U \cap U_1)$ is open in \tilde{U} . Likewise, $AC_{\mathcal{E}}([0, 1], U \cap U_1)$ is open in $\tilde{U}_1 := AC_{\mathcal{E}}([0, 1], U_1)$, endowed with the $C_{\mathbb{K}}^r$ -manifold structure making

$$\tilde{\phi}_1 := AC_{\mathcal{E}}([0, 1], \phi_1): AC_{\mathcal{E}}([0, 1], U_1) \rightarrow AC_{\mathcal{E}}([0, 1], V_1)$$

a global chart for \tilde{U}_1 . Write \tilde{G}_1 for $AC_{\mathcal{E}}([0, 1], G)$, endowed with the unique $C_{\mathbb{K}}^r$ -Lie group structure making \tilde{U}_1 an open submanifold. Then $\text{id}: \tilde{G} \rightarrow \tilde{G}_1$ is a group homomorphism which is $C_{\mathbb{K}}^r$ on the open identity neighbourhood $\tilde{U} \cap \tilde{U}_1$ (as we presently verify) and hence $C_{\mathbb{K}}^r$. Likewise, $\text{id}: \tilde{G}_1 \rightarrow \tilde{G}$ is $C_{\mathbb{K}}^r$, and thus $\tilde{G} = \tilde{G}_1$ as a $C_{\mathbb{K}}^r$ -Lie group. Here, we used that

$$\text{id}|_{\tilde{U} \cap \tilde{U}_1} = (\tilde{\phi}_1)^{-1} \circ AC_{\mathcal{E}}([0, 1], \phi_1 \circ \phi^{-1}|_{\phi(\tilde{U} \cap \tilde{U}_1)}) \circ \tilde{\phi}|_{\tilde{U} \cap \tilde{U}_1}$$

is $C_{\mathbb{K}}^r$. □

Remark 4.3 In the situation of Proposition 4.2, the inclusion map

$$\varepsilon: AC_{\mathcal{E}}([0, 1], G) \rightarrow C([0, 1], G)$$

is a group homomorphism and $C_{\mathbb{K}}^r$.

In fact, for ϕ , $\tilde{\phi}$ and \tilde{U} as in the preceding proof, the map

$$C([0, 1], \phi): C([0, 1], U) \rightarrow C([0, 1], V) \subseteq C([0, 1], E)$$

is a chart for $C([0, 1], G)$. The inclusion map ε is $C_{\mathbb{K}}^r$ on the open identity neighbourhood \tilde{U} since

$$C([0, 1], \phi) \circ \varepsilon|_{\tilde{U}} \circ (\tilde{\phi})^{-1}$$

is the inclusion map $AC_{\mathcal{E}}([0, 1], V) \rightarrow C([0, 1], V)$, which is a restriction of the continuous linear inclusion map $AC_{\mathcal{E}}([0, 1], E) \rightarrow C([0, 1], E)$ and hence $C_{\mathbb{K}}^r$. Since ε is a group homomorphism, it follows that ε is $C_{\mathbb{K}}^r$.

Remark 4.4 It is well known that the evaluation map $\text{ev}_t: C([0, 1], G) \rightarrow G$, $\eta \mapsto \eta(t)$ is a group homomorphism and $C_{\mathbb{K}}^r$ for each $t \in [0, 1]$. By the previous remark, also the evaluation map $\text{ev}_t \circ \varepsilon: AC_{\mathcal{E}}([0, 1], G) \rightarrow G$, $\eta \mapsto \eta(t)$ is a group homomorphism and $C_{\mathbb{K}}^r$, for each $t \in [0, 1]$.

4.5 A local $C_{\mathbb{K}}^r$ -Lie group is a quintuple (U, e, D_U, m, j) , where U is a C_K^r manifold modelled on a locally convex topological \mathbb{K} -vector space E , $e \in U$, D_U an open subset of $U \times U$ such that $(U \times \{e\}) \cup (\{e\} \times U) \subseteq D_U$ and

$$m: D_U \rightarrow U, \quad (x, y) \mapsto m(x, y) =: xy$$

and $j: U \rightarrow U$, $x \mapsto j(x) =: x^{-1}$ are $C_{\mathbb{K}}^r$ -maps satisfying axioms as in [38] or [29]. Then $T_e(U)$ is a Lie algebra.

For example, every open identity symmetric²² neighbourhood U in a $C_{\mathbb{K}}^r$ -Lie group G is a local $C_{\mathbb{K}}^r$ -Lie group with $D_U := \{(x, y) \in U \times U: xy \in U\}$.

Definition 4.6 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) which satisfies the locality axiom (Loc) and such that smooth functions act smoothly

²²That is, $U = U^{-1}$.

on $AC_{\mathcal{E}}$. Let (U, e, D_U, m, j) be a local $C_{\mathbb{K}}^r$ -Lie group which is modelled on a Fréchet space (resp., on a sequentially complete (FEP)-space, resp., on an integral complete locally convex space) E over \mathbb{K} and admits a global chart $\phi: U \rightarrow V \subseteq E$. We then consider $\tilde{U} := AC_{\mathcal{E}}([a, b], U)$ as a local $C_{\mathbb{K}}^r$ -Lie group with the global chart $AC_{\mathcal{E}}([a, b], \phi)$,

$$D_{\tilde{U}} := AC_{\mathcal{E}}([a, b], D_U)$$

(identifying $AC_{\mathcal{E}}([a, b], E^2)$ with $AC([a, b], E)^2$), multiplication $AC_{\mathcal{E}}([a, b], m)$ and the inversion map $AC_{\mathcal{E}}([a, b], j)$.

Definition 4.7 If $(U_1, e_1, D_{U_1}, m_1, j_1)$ and $(U_2, e_2, D_{U_2}, m_2, j_2)$ are local $C_{\mathbb{K}}^r$ -Lie groups, then we call a map $f: U_1 \rightarrow U_2$ a *local group homomorphism* if $(f \times f)(D_{U_1}) \subseteq D_{U_2}$, $m_2 \circ (f \times f)|_{D_{U_1}} = f \circ m_1$ and $j_2 \circ f = f \circ j_1$. If, moreover, f is $C_{\mathbb{K}}^r$, we call f a $C_{\mathbb{K}}^r$ -*homomorphism between local $C_{\mathbb{K}}^r$ -Lie groups*.

Lemma 4.8 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) which satisfies the locality axiom and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let E be Fréchet space (resp., sequentially complete (FEP)-space, resp., integral complete locally convex space) and G be a $C_{\mathbb{K}}^r$ -Lie group (or a local $C_{\mathbb{K}}^r$ -Lie group admitting a global chart) modelled on E . Then the following holds:

- (a) If a, b, α, β, c and d are real numbers with $a \leq \alpha < \beta \leq b$ and $c < d$, let $f: [c, d] \rightarrow [a, b]$ be as in (17). Then $\eta \circ f \in AC_{\mathcal{E}}([c, d], G)$ for each $\eta \in AC_{\mathcal{E}}([a, b], G)$ and the (local) group homomorphism

$$AC_{\mathcal{E}}(f, G): AC_{\mathcal{E}}([a, b], G) \rightarrow AC_{\mathcal{E}}([c, d], G), \quad \eta \mapsto \eta \circ f$$

is $C_{\mathbb{K}}^r$.

- (b) Let $n \in \mathbb{N}$ and $a = t_0 < t_1 < \dots < t_n = b$ be real numbers. If G is a Lie group, then the map

$$\Phi: AC_{\mathcal{E}}([a, b], G) \rightarrow \prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], G), \quad \eta \mapsto (\eta|_{[t_{j-1}, t_j]})_{j \in \{1, \dots, n\}}$$

is a $C_{\mathbb{K}}^r$ -homomorphism. Moreover, Φ is a $C_{\mathbb{K}}^r$ -diffeomorphism onto a $C_{\mathbb{K}}^r$ -submanifold of $\prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], G)$. If G is a local $C_{\mathbb{K}}^r$ -Lie group admitting a global chart, then Φ is a $C_{\mathbb{K}}^r$ -homomorphism between (local) $C_{\mathbb{K}}^r$ -Lie groups and a $C_{\mathbb{K}}^r$ -diffeomorphism onto a $C_{\mathbb{K}}^r$ -submanifold of $\prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], G)$.

(c) For each $t_0 \in [a, b]$,

$$\text{ev}_{t_0}: AC_{\mathcal{E}}([0, 1], G) \rightarrow G, \quad \eta \mapsto \eta(t_0)$$

is a $C_{\mathbb{K}}^r$ -homomorphism. Identifying the Lie algebra \mathfrak{h} of $AC_{\mathcal{E}}([a, b], G)$ with $AC_{\mathcal{E}}([a, b], \mathfrak{g})$ by means of $d(AC_{\mathcal{E}}([a, b], \phi))|_{\mathfrak{h}}$ (where $\phi: U_{\phi} \rightarrow V_{\phi}$ is a chart of G with $e \in U_{\phi}$), the tangent map $L(\text{ev}_{t_0})$ is the point evaluation $\varepsilon_{t_0}: AC_{\mathcal{E}}([a, b], \mathfrak{g}) \rightarrow \mathfrak{g}$, $\eta \mapsto \eta(t_0)$.

Proof. (a) Using Lemma 3.9, we see that the map

$$AC_{\mathcal{E}}(f, E): AC_{\mathcal{E}}([a, b], E) \rightarrow AC_{\mathcal{E}}([c, d], E), \quad \eta \mapsto \eta \circ f \quad (22)$$

is continuous linear (since $\mathcal{E}(f, E)$ is continuous linear by (B2) and also $C(f, E)$ is continuous linear, see, e.g., [20] or [38]). Let $\eta \in AC_{\mathcal{E}}([a, b], G)$. If $t \in [c, d]$, then there exist $\varepsilon > 0$ and a chart $\psi: U_{\psi} \rightarrow V_{\psi}$ for G such that $\eta([f(t) - \varepsilon, f(t) + \varepsilon] \cap [a, b]) \subseteq U_{\psi}$ and

$$\psi \circ \eta|_{[f(t) - \varepsilon, f(t) + \varepsilon] \cap [a, b]} \in AC_{\mathcal{E}}([f(t) - \varepsilon, f(t) + \varepsilon] \cap [a, b], E).$$

We have $f([t - \delta, t + \delta] \cap [c, d]) \subseteq [f(t) - \varepsilon, f(t) + \varepsilon] \cap [a, b]$ for some $\delta > 0$. Write f_t for the restriction of f to a map $[t - \delta, t + \delta] \cap [c, d] \rightarrow [f(t) - \varepsilon, f(t) + \varepsilon]$. Then

$$\psi \circ (\eta \circ f)|_{[t - \delta, t + \delta] \cap [c, d]} = (\psi \circ \eta|_{[f(t) - \varepsilon, f(t) + \varepsilon] \cap [a, b]}) \circ f_t$$

is an element of $AC_{\mathcal{E}}([t - \delta, t + \delta] \cap [c, d], E)$ and hence $\eta \circ f \in AC_{\mathcal{E}}([c, d], G)$, by Remark 3.22.

If $\phi: U \rightarrow V \subseteq E$ is a chart for G defined on an open symmetric identity neighbourhood $U \subseteq G$, then $AC_{\mathcal{E}}([a, b], \phi)$ is chart for $AC_{\mathcal{E}}([a, b], G)$ around e and $AC_{\mathcal{E}}([c, d], \phi)$ is chart for $AC_{\mathcal{E}}([c, d], G)$ around e . Since

$$AC_{\mathcal{E}}([c, d], \phi) \circ AC_{\mathcal{E}}(f, G) \circ AC_{\mathcal{E}}([a, b], \phi)^{-1} = AC_{\mathcal{E}}(f, E)|_{AC_{\mathcal{E}}([a, b], V)}$$

is $C_{\mathbb{K}}^r$, the (local) group homomorphism $AC_{\mathcal{E}}(f, G)$ is $C_{\mathbb{K}}^r$.

(b) The image of Φ is the set

$$\{ \{(\eta_j)_{j \in \{1, \dots, n\}} \in \prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], G) : (\forall j \in \{2, \dots, n\}) \eta_{j-1}(t_j) = \eta_j(t_j) \} \}.$$

Let $\phi: U \rightarrow V \subseteq E$ be a chart for G such that $U \subseteq G$ is a symmetric open identity neighbourhood. Then

$$\psi := \prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], \phi): \prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], U) \rightarrow \prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], V)$$

is a chart for $\prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], G)$. Since

$$F := \{(\eta_j)_{j \in \{1, \dots, n\}} \in \prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], E) : (\forall j \in \{2, \dots, n\}) \eta_{j-1}(t_j) = \eta_j(t_j)\}$$

is a closed vector subspace of $\prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], E)$ and ψ takes

$$\text{im}(\Phi) \cap \prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], U)$$

onto $F \cap \prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], V)$, we have found a submanifold chart for $\text{im}(\Phi)$ around e . If G is a local Lie group with a global chart, then, by the preceding, $\text{im}(\phi)$ is a $C_{\mathbb{K}}^r$ -submanifold of the direct product, if we choose ϕ as the global chart. If G is a Lie group, then $\text{im}(\Phi)$ is a subgroup and translates of ψ provide submanifold charts around each point in $\text{im}(\Phi)$, whence $\text{im}(\Phi)$ is a submanifold of the product modelled on F . In either case, since $\psi|_{\text{im}(\Phi)} \circ \Phi \circ AC_{\mathcal{E}}([a, b], \phi^{-1})$ is the restriction to the open set $AC_{\mathcal{E}}([a, b], V)$ of the isomorphism of topological vector spaces

$$AC_{\mathcal{E}}([a, b], E) \rightarrow F, \quad \eta \mapsto (\eta|_{[t_{j-1}, t_j]})_{j \in \{1, \dots, n\}},$$

we deduce that Φ is a $C_{\mathbb{K}}^r$ -diffeomorphism from $AC_{\mathcal{E}}([a, b], G)$ to the $C_{\mathbb{K}}^r$ -submanifold $\text{im}(\Phi)$.

(c) Follows from $\phi \circ \text{ev}_{t_0} \circ AC_{\mathcal{E}}([a, b], \phi)^{-1} = \varepsilon_{t_0}$. \square

Lemma 4.9 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) which satisfies the locality axiom and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let E be Fréchet space (resp., sequentially complete (FEP)-space, resp., integral complete locally convex space) and G be a $C_{\mathbb{K}}^r$ -Lie group modelled on E . Let $a < b$ be real numbers. Then $AC_{\mathcal{E}}([a, b], G)_* := \{\eta \in AC_{\mathcal{E}}([a, b], G) : \eta(a) = e\}$ is a $C_{\mathbb{K}}^r$ -Lie subgroup of $AC_{\mathcal{E}}([a, b], G)$ and the map*

$$\Psi: AC_{\mathcal{E}}([0, 1], G)_* \times G \rightarrow AC_{\mathcal{E}}([0, 1], G), \quad (\eta, g) \mapsto (t \mapsto \eta(t)g)$$

is a $C_{\mathbb{K}}^r$ -diffeomorphism.

Proof. Let $U \subseteq G$ be a symmetric open identity neighbourhood on which a chart $\phi: U \rightarrow V \subseteq E$ of G is defined. Then $AC_{\mathcal{E}}([a, b], \phi)$ is chart of $AC_{\mathcal{E}}([a, b], G)$ which maps the set

$$AC_{\mathcal{E}}([a, b], U) \cap AC_{\mathcal{E}}([a, b], G)_*$$

onto $AC_{\mathcal{E}}([a, b], V) \cap AC_{\mathcal{E}}([a, b], E)_*$. Hence $AC_{\mathcal{E}}([a, b], G)_*$ is a $C_{\mathbb{K}}^r$ -submanifold of $AC_{\mathcal{E}}([a, b], G)$ modelled on $AC_{\mathcal{E}}([a, b], E)_*$.

The group homomorphism $j_G: G \rightarrow AC_{\mathcal{E}}([a, b], G)$, $j(g)(t) := g$ is $C_{\mathbb{K}}^r$ since, for each chart $\phi: U \rightarrow V \subseteq E$ of G with $e \in U$, we have that

$$AC_{\mathcal{E}}([a, b], \phi) \circ j_G \circ \phi^{-1} = j_E|_V$$

with the linear map $j_E: E \rightarrow AC_{\mathcal{E}}([a, b], E)$, $j_E(v)(t) := v$. The linear map j_E is continuous (as $(\Phi \circ j_E)(v) = (v, 0)$ for Φ as in Definition 3.6 with $t_0 := a$). Thus j_G is $C_{\mathbb{K}}^r$ and hence Ψ is $C_{\mathbb{K}}^r$, since the group multiplication μ of $AC_{\mathcal{E}}([a, b], G)$ is $C_{\mathbb{K}}^r$ and $\Psi(\eta, g) = \eta j_G(g) = \mu(\eta, j_G(g))$. The evaluation map $\text{ev}_a: AC_{\mathcal{E}}([a, b], G) \rightarrow G$, $\eta \mapsto \eta(a)$ is $C_{\mathbb{K}}^r$ by Lemma 4.8 (c). The map

$$\Psi^{-1}: AC_{\mathcal{E}}([a, b], G) \rightarrow AC_{\mathcal{E}}([a, b], G)_* \times G, \quad \eta \mapsto (\mu((j_G(\text{ev}_a(\eta)))^{-1}, \eta), \text{ev}_a(\eta))$$

is $C_{\mathbb{K}}^r$ as a map to $AC_{\mathcal{E}}([a, b], G) \times G$ and hence also $C_{\mathbb{K}}^r$ as a map to the $C_{\mathbb{K}}^r$ -submanifold $AC_{\mathcal{E}}([a, b], G)_* \times G$. Thus Ψ is a $C_{\mathbb{K}}^r$ -diffeomorphism. \square

Definition 4.10 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex space). We say that \mathcal{E} satisfies the *embedding axiom* if the following holds:

- (E) For real numbers $a < b$ and each Fréchet space (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex space) E and closed vector subspace F , we have

$$AC_{\mathcal{E}}([a, b], F) = \{\eta \in AC_{\mathcal{E}}([a, b], E) : \eta([a, b]) \subseteq F\}, \quad (23)$$

and the linear map

$$\mathcal{E}([a, b], j): \mathcal{E}([a, b], F) \rightarrow \mathcal{E}([a, b], E), \quad [\gamma] \mapsto [j \circ \gamma]$$

induced by the inclusion map $j: F \rightarrow E$ is a topological embedding.

As the evaluation maps $AC_{\mathcal{E}}([a, b], E) \rightarrow E$, $\eta \mapsto \eta(t)$ are continuous, (23) implies that $AC_{\mathcal{E}}([a, b], F)$ is a *closed* vector subspace of $AC_{\mathcal{E}}([a, b], E)$.

Remark 4.11 Lemma 1.47 implies that L^p as a bifunctor on Fréchet spaces (for any $p \in [1, \infty]$) and L_{rc}^{∞} as a bifunctor on integral complete locally convex spaces satisfy the embedding axiom.²³

Lemma 4.12 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex space) which satisfies the locality axiom and such that smooth functions act on $AC_{\mathcal{E}}$. Let M be a $C_{\mathbb{K}}^r$ -manifold modelled on such a space E and $N \subseteq M$ be a $C_{\mathbb{K}}^r$ -submanifold modelled on a closed vector subspace $F \subseteq E$. Let $a < b$ be real numbers. If F is complemented in E or \mathcal{E} satisfies the embedding axiom, then a map $\eta: [a, b] \rightarrow N$ is in $AC_{\mathcal{E}}([a, b], N)$ if and only if it is in $AC_{\mathcal{E}}([a, b], M)$.*

Proof. Let $j: N \rightarrow M$ be the inclusion map. If $\eta \in AC_{\mathcal{E}}([a, b], N)$, then $j \circ \eta \in AC_{\mathcal{E}}([a, b], M)$, by Lemma 3.24.

If, conversely, $\eta \in AC_{\mathcal{E}}([a, b], M)$ and $t \in [a, b]$, we find a chart $\phi: U \rightarrow V \subseteq E$ of M such that $\phi(U \cap N) = V \cap F$. There is $\delta > 0$ such that $\eta([a, b] \cap [t - \delta, t + \delta]) \subseteq U$. Then $\psi := \phi|_{U \cap N}: U \cap N \rightarrow V \cap F$ is chart for N , and

$$\psi \circ \eta|_{[a, b] \cap [t - \delta, t + \delta]}$$

is a mapping to F which is in $AC_{\mathcal{E}}([a, b] \cap [t - \delta, t + \delta], E)$ (as it coincides with $\phi \circ \eta|_{[a, b] \cap [t - \delta, t + \delta]}$ as a mapping to E). If \mathcal{E} satisfies the embedding axiom, this implies that

$$\psi \circ \eta|_{[a, b] \cap [t - \delta, t + \delta]} \in AC_{\mathcal{E}}([a, b] \cap [t - \delta, t + \delta], F). \quad (24)$$

If F is complemented in E , then we find a continuous linear map $\lambda: E \rightarrow F$ such that $\lambda|_F = \text{id}_F$. Again,

$$\begin{aligned} \psi \circ \eta|_{[a, b] \cap [t - \delta, t + \delta]} &= \lambda \circ \phi \circ \eta|_{[a, b] \cap [t - \delta, t + \delta]} \\ &= AC_{\mathcal{E}}([a, b] \cap [t - \delta, t + \delta], \lambda)(\phi \circ \eta|_{[a, b] \cap [t - \delta, t + \delta]}) \\ &\in AC_{\mathcal{E}}([a, b] \cap [t - \delta, t + \delta], F). \end{aligned}$$

Hence $\eta \in AC_{\mathcal{E}}([a, b], N)$, by Remark 3.22. □

²³For R , the author would not expect this. For L^p as a bifunctor on sequentially complete (FEP)-spaces, the author did not succeed to prove the property.

Lemma 4.13 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex space) which satisfies the locality axiom and such that smooth functions act on $AC_{\mathcal{E}}$. Let G be a $C_{\mathbb{K}}^r$ -Lie group modelled on such a space E and $H \subseteq G$ be a $C_{\mathbb{K}}^r$ -Lie subgroup modelled on a closed vector subspace $F \subseteq E$. Let $a < b$ be real numbers. If F is complemented in E or \mathcal{E} satisfies the embedding axiom, then $AC_{\mathcal{E}}([a, b], H)$ is a $C_{\mathbb{K}}^r$ -submanifold of $AC_{\mathcal{E}}([a, b], G)$.*

Proof. Let $\phi: U \rightarrow V \subseteq E$ be a chart for G defined on a symmetric open identity neighbourhood $U \subseteq G$ such that

$$\phi(U \cap H) = V \cap F. \quad (25)$$

Since F is complemented in E (in which case $AC_{\mathcal{E}}([a, b],) = AC_{\mathcal{E}}([a, b], F) \oplus AC_{\mathcal{E}}([a, b], Y)$ if $E = F \oplus Y$) or \mathcal{E} satisfies the embedding axiom, we have that

$$AC_{\mathcal{E}}([a, b], F) = \{\eta \in AC_{\mathcal{E}}([a, b], E) : \eta([a, b]) \subseteq F\} \quad (26)$$

is a closed vector subspace of $AC_{\mathcal{E}}([a, b], E)$ and that the inclusion map $AC_{\mathcal{E}}([a, b], F) \rightarrow AC_{\mathcal{E}}([a, b], E)$ is a topological embedding. As the chart $AC_{\mathcal{E}}([a, b], \phi)$ of $AC_{\mathcal{E}}([a, b], G)$ takes

$$AC_{\mathcal{E}}([a, b], U) \cap AC_{\mathcal{E}}([a, b], H)$$

onto the set $AC_{\mathcal{E}}([a, b], V) \cap AC_{\mathcal{E}}([a, b], F)$ by (25) and (26), we deduce that the subgroup $AC_{\mathcal{E}}([a, b], H)$ is a $C_{\mathbb{K}}^r$ -submanifold of $AC_{\mathcal{E}}([a, b], G)$ modelled on $AC_{\mathcal{E}}([a, b], F)$. Since $AC_{\mathcal{E}}([a, b], \phi)$ restricts to the chart $AC_{\mathcal{E}}([a, b], \phi|_{U \cap H}^{V \cap F})$ of $AC_{\mathcal{E}}([a, b], H)$, the given $C_{\mathbb{K}}^r$ -Lie group structure on $AC_{\mathcal{E}}([a, b], H)$ coincides with the manifold structure as a $C_{\mathbb{K}}^r$ -submanifold of $AC_{\mathcal{E}}([a, b], G)$. \square

Remark 4.14 Consider a strict (LF)-space E and a vector subspace $F \subseteq E$ which is a Fréchet space in the induced topology. Then the conclusion of Lemma 4.12 remains valid for $\mathcal{E} = L^1$ because (24) is satisfied by Lemma 1.47. As a consequence, also the conclusion of Lemma 4.13 remains valid for $\mathcal{E} = L^1$ whenever E is a strict (LF)-space and $F \subseteq E$ a Fréchet subspace.

5 \mathcal{E} -regularity and local \mathcal{E} -regularity

Definition 5.1 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, and such that smooth functions act on $AC_{\mathcal{E}}$. Let E be a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space) and M be a smooth manifold modelled on E . Let $a < b$ be real numbers and $\eta \in AC_{\mathcal{E}}([a, b], M)$. Let $a = t_0 < t_1 < \dots < t_n = b$ such that $\eta([t_{j-1}, t_j]) \subseteq U_j$ for a chart $\phi_j: U_j \rightarrow V_j \subseteq E$ of M . Then

$$\eta_j := \phi_j \circ \eta|_{[t_{j-1}, t_j]} \in AC_{\mathcal{E}}([t_{j-1}, t_j], E)$$

for all $j \in \{1, \dots, n\}$, enabling us to define

$$\eta'_j \in \mathcal{E}([a, b], E).$$

Write $\eta'_j = [\gamma_j]$ with $\gamma_j \in \mathcal{L}^1([t_{j-1}, t_j], E)$ (resp., $\gamma_j \in \mathcal{L}_{rc}^{\infty}([t_{j-1}, t_j], E)$). We define

$$\gamma: [a, b] \rightarrow TM$$

via $\gamma(t) := T(\phi_j)^{-1}(\eta_j(t), \gamma_j(t))$ if $t \in [t_{j-1}, t_j[$ with $j \in \{1, \dots, n\}$, and $\gamma(b) = T(\phi_n)^{-1}(\eta_n(b))$. Then γ is measurable and we define

$$\dot{\eta} := [\gamma].$$

Remark 5.2 (a) If $\pi_{TM}: TM \rightarrow M$ is the bundle projection taking $v \in T_x M$ to x , then $\pi_{TM} \circ \gamma = \eta$ is a continuous map (this property enters into Lemma 5.5).

(b) If $f: M \rightarrow N$ is a smooth map between smooth manifolds in the preceding situation, then $f \circ \eta \in AC_{\mathcal{E}}([a, b], N)$ for each $\eta \in AC_{\mathcal{E}}([a, b], M)$ and

$$(f \circ \eta) \cdot = [Tf \circ \gamma] \quad \text{if } \dot{\eta} = [\gamma]; \tag{27}$$

this follows from Lemma 3.24 and Remark 3.19.

5.3 Let G be a Lie group, with multiplication $m_G: G \times G \rightarrow G$ and inversion $j_G: G \rightarrow G$. Let TG be the tangent bundle, considered as a Lie group with multiplication $T(m_G)$ (identifying $T(G \times G)$ with $TG \times TG$) and inversion

$T(i_G)$. We identify $g \in G$ with $0_g \in T_g(G)$. Then $0_e \in T_e(G) =: L(G) =: \mathfrak{g}$ is the neutral element for TG . If $v \in T_g G$ and $w \in T_h(G)$ with $g, h \in G$, then

$$vw = gw + vh, \quad (28)$$

where $gw = T_h \lambda_g(w)$ and $vh = T_g \rho_h(v)$ with the left translation $\lambda_g: G \rightarrow G$, $x \mapsto gx$ and the right translation $\rho_h: G \rightarrow G$, $x \mapsto xh$. In the following, we consider the smooth \mathfrak{g} -valued 1-forms

$$\omega_\ell: TG \rightarrow \mathfrak{g}, \quad v \mapsto (\pi_{TG}(v))^{-1}(v)$$

and

$$\omega_r: TG \rightarrow \mathfrak{g}, \quad v \mapsto v(\pi_{TG}(v))^{-1}.$$

Likewise if G is a local Lie group.

Lemma 5.4 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces). Assume that \mathcal{E} satisfies the locality axiom and that smooth functions act on $AC_{\mathcal{E}}$. Let G be a Lie group (or local Lie group) modelled on a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space) E , and $a < b$. Let $\eta, \eta_1, \eta_2 \in AC_{\mathcal{E}}([a, b], G)$ and write η^{-1} for the map $[a, b] \rightarrow G$, $t \mapsto (\eta(t))^{-1}$. Write $\dot{\eta} = [\gamma]$, $\dot{\eta}_1 = [\gamma_1]$, and $\dot{\eta}_2 = [\gamma_2]$ with measurable functions $\gamma, \gamma_1, \gamma_2: [a, b] \rightarrow TG$. Then*

$$(\eta_1 \eta_2)^\cdot = [t \mapsto \gamma_1(t) \eta_2(t) + \eta_1(t) \gamma_2(t)] \text{ and} \quad (29)$$

$$(\eta^{-1})^\cdot = [t \mapsto -\eta(t)^{-1} \gamma(t) \eta(t)^{-1}], \quad (30)$$

assuming that $\eta_1 \eta_2$ is defined in the case of a local Lie group G .

Proof. This follows from (28) and Remark 3.19. \square

Lemma 5.5 *Let G be a Lie group (or local Lie group) with Lie algebra \mathfrak{g} and $\gamma: [a, b] \rightarrow TG$ be a map such that $\pi_{TG} \circ \gamma$ is continuous. Then γ is measurable if and only if $\omega_r \circ \gamma: [a, b] \rightarrow \mathfrak{g}$ is measurable, if and only if $\omega_\ell \circ \gamma: [a, b] \rightarrow \mathfrak{g}$ is measurable.*

Proof. If γ is measurable, then also $\omega_r \circ \gamma$ and $\omega_\ell \circ \gamma$, as ω_r and ω_ℓ are smooth mappings, hence continuous and hence Borel measurable. The map

$\sigma: G \times \mathfrak{g} \rightarrow TG$, $(g, v) \mapsto gv$ obtained from multiplication in TG is smooth and hence Borel measurable. If $\omega_\ell(\gamma)$ is measurable and $\eta := \pi_{TG} \circ \gamma$ is continuous, then $\eta([a, b]) \subseteq G$ is compact and metrizable (see Lemma 1.10). By 1.6 (e) and (f), the map $(\eta, \omega_\ell \circ \gamma): [a, b] \rightarrow G \times TG$ is Borel measurable. Hence so is $\gamma = \sigma \circ (\eta, \omega_\ell \circ \gamma)$. If $\omega_r \circ \gamma$ is measurable, we can argue in the same way, using $\sigma: G \times \mathfrak{g} \rightarrow TG$, $\sigma(g, v) := vg$ instead. \square

Definition 5.6 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) such that \mathcal{E} satisfies the locality axiom (Loc) and smooth functions act on $AC_{\mathcal{E}}$. Let E be a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space) and G be a Lie group (or local Lie group) modelled on E . If $a < b$ are real numbers and $\eta \in AC_{\mathcal{E}}([a, b], G)$, we define the *left logarithmic derivative* of η via

$$\delta(\eta) := \delta^\ell(\eta) := [\omega_\ell \circ \gamma],$$

where $\dot{\eta} = [\gamma]$ with a measurable function $\gamma: [a, b] \rightarrow TG$. Similarly, we define the *right logarithmic derivative* of η via

$$\delta^r(\eta) := [\omega_r \circ \gamma],$$

where $\dot{\eta} = [\gamma]$ with a measurable function $\gamma: [a, b] \rightarrow TG$.

Definition 5.7 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces). We say that \mathcal{E} satisfies the *pushforward axioms* if the following holds:

(P1) Let $a < b$ be real numbers, E_1 be a locally convex space, $V \subseteq E_1$ be an open subset and

$$f: V \times E_2 \rightarrow F$$

be a continuous map which is linear in the second argument, where E_2 and F are Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces). Then

$$\tilde{f}(\eta, [\gamma]) := [f \circ (\eta, \gamma)] \in \mathcal{E}([a, b], F)$$

for all $[\gamma] \in \mathcal{E}([a, b], E_2)$.

(P2) If f is smooth in the situation of (P1), then also the map

$$\tilde{f}: C([a, b], V) \times \mathcal{E}([a, b], E_2) \rightarrow \mathcal{E}([a, b], F)$$

is smooth.

By Lemma 2.1 and Propositions 2.2 and 2.3, we have:

Lemma 5.8 *The bifunctors L^p on Fréchet spaces (or sequentially complete (FEP)-space) satisfy the pushforward axioms for all $p \in [1, \infty]$. Moreover, the bifunctors L_{rc}^∞ and R on integral complete locally convex spaces satisfy the pushforward axioms. \square*

Lemma 5.9 *In the situation of Definition 5.7 (P2), we have*

$$d\tilde{f}((\eta_1, [\gamma_1]), (\eta_2, [\gamma_2])) = [df \circ (\eta_1, \gamma_1, \eta_2, \gamma_2)] \quad (31)$$

for all $\eta_1 \in C([a, b], V)$, $\eta_2 \in C([a, b], E_1)$ and $[\gamma_1], [\gamma_2] \in \mathcal{E}([a, b], E_2)$.

Proof. In the case of a bifunctor on Fréchet spaces or sequentially complete (FEP)-spaces, set $\mathcal{F} := L^1$; in the case of a bifunctor on integral complete locally convex spaces, set $\mathcal{F} := L_{rc}^\infty$. By Propositions 2.2 and 2.3, the map

$$g: C([a, b], V) \times \mathcal{F}([a, b], E_2) \rightarrow \mathcal{F}([a, b], F), \quad (\eta, [\gamma]) \mapsto [f \circ (\eta, \gamma)]$$

is smooth; moreover,

$$dg((\eta_1, [\gamma_1]), (\eta_2, [\gamma_2])) = [df \circ (\eta_1, \gamma_1, \eta_2, \gamma_2)]$$

for all $\eta_1 \in C([a, b], V)$, $\eta_2 \in C([a, b], E_1)$ and $[\gamma_1], [\gamma_2] \in \mathcal{F}([a, b], E_2)$, by (15). Let $j_F: \mathcal{E}([a, b], F) \rightarrow \mathcal{F}([a, b], F)$ be the inclusion map and define j_{E_2} analogously. Let j_{E_1} be the identity map of $C([a, b], E_1)$. Then

$$j_F \circ \tilde{f} = g \circ (j_{E_1}|_{C([a, b], V)} \times j_{E_2})$$

and thus, using the Chain Rule,

$$j_F \circ d\tilde{f} = dg \circ (j_{E_1}|_{C([a, b], V)} \times j_{E_2} \times j_{E_1} \times j_{E_2}).$$

Hence, for $\eta_1 \in C([a, b], V)$, $\eta_2 \in C([a, b], E_1)$ and $[\gamma_1], [\gamma_2] \in \mathcal{E}([a, b], E_2)$,

$$\begin{aligned} d\tilde{f}(\eta_1, [\gamma_1], \eta_2, [\gamma_2]) &= j_F(d\tilde{f}(\eta_1, [\gamma_1], \eta_2, [\gamma_2])) = dg(\eta_1, [\gamma_1], \eta_2, [\gamma_2]) \\ &= [df \circ (\eta_1, \gamma_1, \eta_2, \gamma_2)], \end{aligned}$$

establishing (31). \square

Lemma 5.10 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom and the pushforward axioms. Let $a < b$ be real numbers, E_2 and F be Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces), M be a manifold modelled on a locally convex space E_1 and*

$$f: M \times E_2 \rightarrow F$$

be a smooth map which is linear in the second argument. Let $\eta \in C([a, b], M)$. Then

$$\tilde{f}(\eta, [\gamma]) := [f \circ (\eta, \gamma)] \in \mathcal{E}([a, b], F)$$

for all $[\gamma] \in \mathcal{E}([a, b], E_2)$, and the map

$$\mathcal{E}([a, b], E_2) \rightarrow \mathcal{E}([a, b], F), \quad [\gamma] \mapsto \tilde{f}(\eta, [\gamma])$$

is continuous linear. If $M = G$ is a Lie group, then

$$\tilde{f}: C([a, b], G) \times \mathcal{E}([a, b], E_2) \rightarrow \mathcal{E}([a, b], F)$$

is smooth.

Proof. Fix $\eta \in C([a, b], M)$. Using a compactness argument, we find $a = t_0 < t_1 < \dots < t_n = b$ such that $\eta([t_{j-1}, t_j]) \subseteq U_j$ for a chart $\phi_j: U_j \rightarrow V_j \subseteq E_1$ of M . Then

$$f_j: V_j \times E_2 \rightarrow F, \quad f_j(x, y) := f(\phi_j^{-1}(x), y)$$

is a smooth map which is linear in its second argument and thus

$$\tilde{f}_j: C([t_{j-1}, t_j], V_j) \times \mathcal{E}([t_{j-1}, t_j], E_2) \rightarrow \mathcal{E}([t_{j-1}, t_j], F), \quad (\sigma, [\tau]) \mapsto [f_j \circ (\sigma, \tau)]$$

is smooth by the pushforward axiom (P2). For $[\gamma] \in \mathcal{E}([a, b], E_2)$, we have $[\gamma|_{[t_{j-1}, t_j]}] \in \mathcal{E}([t_{j-1}, t_j], E_2)$ by (B2) and hence

$$[f \circ (\eta, \gamma)|_{[t_{j-1}, t_j]}] = \tilde{f}_j(\phi_j \circ \eta|_{[t_{j-1}, t_j]}, \gamma|_{[t_{j-1}, t_j]}) \in \mathcal{E}([t_{j-1}, t_j], F)$$

for all $j \in \{1, \dots, n\}$. By the locality axiom, we get

$$\tilde{f}(\eta, \gamma) = [f \circ (\eta, \gamma)] \in \mathcal{E}([a, b], F).$$

Again by the locality axiom, the linear map $\tilde{f}(\eta, \cdot)$ will be continuous if we can show that the map

$$\mathcal{E}([a, b], E_2) \rightarrow \mathcal{E}([t_{j-1}, t_j], F), \quad [\gamma] \mapsto [f \circ (\eta, \gamma)|_{[t_{j-1}, t_j]}]$$

is continuous for each $j \in \{1, \dots, n\}$. But this map is the composition of the continuous map \tilde{f}_j and the map

$$\mathcal{E}([a, b], E_2) \rightarrow C([t_{j-1}, t_j], V) \times \mathcal{E}([t_{j-1}, t_j], E_2),$$

$[\gamma] \mapsto (\phi_j \circ \eta|_{[t_{j-1}, t_j]}, [\gamma|_{[t_{j-1}, t_j]}])$, which is continuous by (B2).

If $M = G$ is a Lie group and $\eta \in C([a, b], G)$, let us show that \tilde{f} is smooth on $P \times \mathcal{E}([a, b], E_2)$ for some open neighbourhood P of η in $C([a, b], G)$. Let $U \subseteq G$ be a symmetric open identity neighbourhood on which a chart $\phi: U \rightarrow V \subseteq E_1$ of G is defined. Let $W \subseteq G$ be an open identity neighbourhood such that $WW \subseteq U$. We may assume that $a = t_0 < \dots < t_n = b$ has been chosen such that

$$\eta([t_{j-1}, t_j]) \subseteq \eta(t_{j-1})W.$$

Then

$$P := \{\zeta \in C([0, 1], G) : (\eta^{-1}\zeta)([a, b]) \subseteq W\}$$

is an open neighbourhood of η in $C([a, b], G)$. For $\zeta \in P$ and $t \in [t_{j-1}, t_j]$, we have

$$\zeta(t) = \eta(t)\eta(t)^{-1}\zeta(t) \in \eta(t_{j-1})WW \subseteq \eta(t_{j-1})U.$$

Now

$$\psi_j: \eta(t_{j-1})C([t_{j-1}, t_j], U) \rightarrow C([t_{j-1}, t_j], V), \quad \tau \mapsto \phi \circ (\eta(t_{j-1})^{-1}\tau)$$

is a chart for $C([t_{j-1}, t_j], G)$ around $\eta|_{[t_{j-1}, t_j]}$ such that

$$\zeta|_{[t_{j-1}, t_j]}$$

is in the domain $\eta(t_{j-1})C([t_{j-1}, t_j], U)$ of ψ_j for each $\eta \in P$. The restriction map

$$\rho_j: C([a, b], G) \rightarrow C([t_{j-1}, t_j], G), \quad \tau \mapsto \tau|_{[t_{j-1}, t_j]}$$

is a smooth group homomorphism (cf. [20]). The map

$$r_j: \mathcal{E}([a, b], E_2) \rightarrow \mathcal{E}([t_{j-1}, t_j], E_2), \quad [\tau] \mapsto [\tau|_{[t_{j-1}, t_j]}]$$

is continuous linear (and hence smooth), by (B2). The map

$$g_j: V \times E_2 \rightarrow F, \quad g_j(x, y) := f(\eta(t_{j-1})\phi^{-1}(x), y)$$

is linear in its second argument and smooth. Hence, by the pushforward axiom (P2), the map

$$\tilde{g}_j: C([t_{j-1}, t_j], V) \times \mathcal{E}([t_{j-1}, t_j], E_2) \rightarrow \mathcal{E}([t_{j-1}, t_j], F), \quad (\sigma, [\tau]) \mapsto [g_j \circ (\sigma, \tau)]$$

is smooth. By the locality axiom, the map \tilde{f} will be smooth on $P \times \mathcal{E}([a, b], E_2)$ if we can show that the map

$$h_j: P \times \mathcal{E}([a, b], E_2) \rightarrow \mathcal{E}([t_{j-1}, t_j], F), \quad (\zeta, [\gamma]) \mapsto [f \circ (\zeta, \gamma)|_{[t_{j-1}, t_j]}]$$

is smooth for all $j \in \{1, \dots, n\}$. But h_j is the map

$$(\zeta, [\gamma]) \mapsto \tilde{g}_j(\psi_j(\rho_j(\zeta)), r_j([\gamma]))$$

and hence h_j is smooth as a composition of smooth maps. \square

Lemma 5.11 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axiom (P1), and such that smooth functions act on $AC_{\mathcal{E}}$. Let G be a Lie group (or local Lie group) modelled on a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space), and $a < b$. If $\eta \in AC_{\mathcal{E}}([a, b], G)$, then $\delta^\ell(\eta), \delta^r(\eta) \in \mathcal{E}([a, b], \mathfrak{g})$.*

Proof. Let E be the modelling space of G . With $M := G$, let $a = t_0 < \dots < t_n = b$, ϕ_j, η_j, γ_j and γ be as in Definition 5.1. For each $j \in \{1, \dots, n\}$,

$$f_j := \omega_\ell|_{TV_j} \circ T\phi_j^{-1}: V_j \times E \rightarrow \mathfrak{g}$$

is a C^∞ -map and linear in its second argument. Now $\eta_j \in AC_{\mathcal{E}}([t_{j-1}, t_j], E)$ and $\eta'_j = [\gamma_j] \in \mathcal{E}([a, b], E)$. By definition,

$$\delta^\ell(\eta) = [\omega_\ell \circ \gamma]$$

where $\omega_\ell(\gamma(t)) = \omega_\ell(T\phi_j^{-1}(\eta_j(t), \gamma_j(t))) = f_j(\eta_j(t), \gamma_j(t))$ for $t \in [t_{j-1}, t_j]$. By the pushforward axiom, $[\omega_\ell \circ \gamma|_{[t_{j-1}, t_j]}] = [f_j \circ (\eta_j, \gamma_j)] \in \mathcal{E}([t_{j-1}, t_j], \mathfrak{g})$. Hence $\delta^\ell(\eta) = [\omega_\ell \circ \gamma] \in \mathcal{E}([a, b], \mathfrak{g})$, by the locality axiom. The proof for $\delta^r(\eta)$ is similar. \square

Lemma 5.12 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces). Assume that \mathcal{E} satisfies the locality axiom (Loc), the pushforward axiom (P1), and that smooth functions act on $AC_{\mathcal{E}}$. Let E be a Fréchet space (resp., a sequentially complete (FEP)-space, an integral complete locally convex space) and G be a Lie group modelled on E . If $a < b$ are real numbers and $\eta, \eta_1, \eta_2 \in AC_{\mathcal{E}}([a, b], G)$, then*

$$\delta^{\ell}(\eta_1 \eta_2^{-1}) = [t \mapsto \text{Ad}(\eta_2(t))(\gamma_1 - \gamma_2)] \quad (32)$$

with $\delta^{\ell}(\eta_j) = [\gamma_j]$ for $j \in \{1, 2\}$ and

$$\delta^r(\eta_1^{-1} \eta_2) = [t \mapsto \text{Ad}(\eta_1(t))^{-1}(\zeta_2 - \zeta_1)] \quad (33)$$

with $\delta^r(\eta_j) = [\zeta_j]$ for $j \in \{1, 2\}$. Also,

$$\delta^{\ell}(\eta_1 \eta_2) = [t \mapsto \text{Ad}(\eta_2(t))^{-1}(\gamma_1(t)) + \gamma_2(t)] \quad (34)$$

and

$$\delta^{\ell}(\eta^{-1}) = -\delta^r(\eta). \quad (35)$$

If $\delta^{\ell}(\eta) = 0$ or $\delta^r(\eta) = 0$, then η is constant. Moreover, $\delta^{\ell}(\eta_1) = \delta^{\ell}(\eta_2)$ if and only if $\eta_2 = g\eta_1$ for some $g \in G$. Likewise, $\delta^r(\eta_1) = \delta^r(\eta_2)$ if and only if $\eta_2 = \eta_1 g$ for some $g \in G$.

If G is a local Lie group modelled on E , then (35) always holds while (32), (33) and (34) hold whenever $\eta_1 \eta_2^{-1}$, $\eta_1^{-1} \eta_2$ and $\eta_1 \eta_2$, respectively, are defined. If $\delta^{\ell}(\eta_1) = \delta^{\ell}(\eta_2)$ (or $\delta^r(\eta_1) = \delta^r(\eta_2)$) and $\eta_1(t_0) = \eta_2(t_0)$ for some $t_0 \in [a, b]$, then $\eta_1 = \eta_2$.

Proof. Assume first that G is a Lie group. (32), (33), (34) and (35) follow immediately from (5.4) and the definition of logarithmic derivatives.

If $\delta^{\ell}(\eta) = 0$, then $\eta'_j = [\gamma_j] = 0$ for all $j \in \{1, \dots, n\}$ in the proof of Lemma 5.11, whence η_j and $\eta|_{[t_{j-1}, t_j]} = \phi_j^{-1} \circ \eta_j$ are constant; thus η is constant.

If $\delta^{\ell}(\eta_1) = \delta^{\ell}(\eta_2)$, then $\delta^{\ell}(\eta_1 \eta_2^{-1}) = 0$ by (32) and thus $\eta_1 \eta_2^{-1}$ is constant, taking the value $g \in G$, say. Thus $\eta_1 = g\eta_2$ and $\eta_2 = g^{-1}\eta_1$. The proof for right logarithmic derivatives is analogous.

If G is a local Lie group, we can establish (32)–(35) as before if all expressions are defined. If $\delta^\ell(\eta_1) = \delta^\ell(\eta_2)$ and $\eta_1(t_0) = \eta_2(t_0)$ for some $t_0 \in [a, b]$, then

$$A := \{t \in [a, b] : \eta_1(t) = \eta_2(t)\}$$

is a non-empty, closed subset of $[a, b]$. If we can show that A is also open, then $A = [a, b]$ (as $[a, b]$ is connected) and hence $\eta_1 = \eta_2$. If $t_1 \in A$, we find $\delta > 0$ such that $\theta(t) := \eta_1(t)\eta_2(t)^{-1}$ is defined for all $t \in [t_1 - \delta, t_1 + \delta] \cap [a, b]$ and

$$\theta(t)\eta_2(t) = \eta_1(t).$$

Then $\delta^\ell(\theta) = 0$ by (32), whence θ is constant (as in the group case). Since $\theta(t_1) = e$, we deduce that $\theta(t) = e$ for all $t \in [t_1 - \delta, t_1 + \delta] \cap [a, b]$, whence $\eta_1(t) = \eta_2(t)$ and $[t_1 - \delta, t_1 + \delta] \cap [a, b] \subseteq A$. The proof for right logarithmic derivatives is similar. \square

Remark 5.13 Identifying measurable functions and their equivalence classes, (32), (33) and (34) can be rewritten as

$$\begin{aligned} \delta^\ell(\eta_1\eta_2^{-1})(t) &= \text{Ad}(\eta_2(t)).(\delta^\ell(\eta_1)(t) - \delta^\ell(\eta_2)(t)), \\ \delta^r(\eta_1^{-1}\eta_2)(t) &= \text{Ad}(\eta_1(t))^{-1}(\delta^r(\eta_2)(t) - \delta^r(\eta_1)(t)) \text{ and} \\ \delta^\ell(\eta_1\eta_2)(t) &= \text{Ad}(\eta_2(t))^{-1}(\delta^\ell(\eta_1)(t)) + \delta^\ell(\eta_2)(t) \end{aligned}$$

for λ_1 -almost all $t \in [a, b]$.

Definition 5.14 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom. Let E be a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space), $W \subseteq \mathbb{R} \times E$ be a subset and $f : W \rightarrow E$ be a map. We say that a continuous function $\eta : I \rightarrow E$ on a non-degenerate interval $I \subseteq \mathbb{R}$ is an *AC \mathcal{E} -Carathéodory solution* to the differential equation

$$y' = f(t, y)$$

if $(t, \eta(t)) \in W$ for all $t \in I$, the map

$$t \mapsto f(t, \eta(t))$$

is in $\mathcal{E}(I, E)$, and

$$(\forall t_1, t_2 \in I) \quad \eta(t_2) - \eta(t_1) = \int_{t_1}^{t_2} f(s, \eta(s)) ds.$$

Or equivalently, if $\eta \in AC_{\mathcal{E}}(I, E)$ with $\text{graph}(\eta) \subseteq W$ and

$$\eta' = [t \mapsto f(t, \eta(t))].$$

If $(t_0, y_0) \in W$ and η is as before with $t_0 \in I$ and $\eta(t_0) = y_0$, then we call η an *AC $_{\mathcal{E}}$ -Carathéodory solution* to the initial value problem²⁴

$$\begin{cases} y' &= f(t, y) \\ y(t_0) &= y_0. \end{cases}$$

Equivalently, $\eta: I \rightarrow E$ is a continuous function on a non-degenerate interval $I \subseteq \mathbb{R}$ with $t_0 \in I$ such that $\eta(t_0) = y_0$, $(t, \eta(t)) \in W$ for all $t \in I$, the map

$$t \mapsto f(t, \eta(t))$$

is in $\mathcal{E}(I, E)$, and

$$(\forall t \in I) \quad \eta(t) = y_0 + \int_{t_0}^t f(s, \eta(s)) ds.$$

Or equivalently, if $\eta \in AC_{\mathcal{E}}(I, E)$ with $\text{graph}(\eta) \subseteq W$ such that $t_0 \in I$, $\eta(t_0) = y_0$ and

$$\eta' = [t \mapsto f(t, \eta(t))].$$

Definition 5.15 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom and such that smooth functions act on $AC_{\mathcal{E}}$. Let E be a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex spaces), M be a smooth manifold modelled on E , $W \subseteq \mathbb{R} \times M$ be a subset and $f: W \rightarrow TM$ be a map such that $f(t, y) \in T_y(M)$ for all $(t, y) \in W$. Let $(t_0, y_0) \in W$. An *AC $_{\mathcal{E}}$ -Carathéodory solution* to

$$\begin{cases} y' &= f(t, y) \\ y(t_0) &= y_0 \end{cases}$$

²⁴Compare [62] for the case that E is a Banach space.

is a map $\eta \in AC_{\mathcal{E}}(I, M)$ on a non-degenerate interval $I \subseteq \mathbb{R}$ with $t_0 \in I$ such that $\text{graph}(\eta) \subseteq W$, $\eta(t_0) = y_0$ and such that, for each $t \in I$, there exists $\varepsilon > 0$ such that $\eta(I \cap]t - \varepsilon, t + \varepsilon[) \subseteq U$ for some chart $\phi: U \rightarrow V \subseteq E$ of M and $\zeta := \phi \circ \eta|_{I \cap]t - \varepsilon, t + \varepsilon[)}$ is an $AC_{\mathcal{E}}$ -Carathéodory solution to $y' = g(t, y)$ with

$$g: (\text{id}_{\mathbb{R}} \times \phi)(W \cap (\mathbb{R} \times U)) \rightarrow E, \quad g(t, y) := d\phi(f(t, \phi^{-1}(y))).$$

Definition 5.16 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex space) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act on $AC_{\mathcal{E}}$. Let G be a Lie group modelled on a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space), with Lie algebra \mathfrak{g} and neutral element e . We say that G is \mathcal{E} -semiregular if for each $\gamma \in \mathcal{E}([0, 1], \mathfrak{g})$, there exists $\eta \in AC_{\mathcal{E}}([0, 1], G)$ such that

$$\delta^\ell(\eta) = \gamma \quad \text{and} \quad \eta(0) = e. \quad (36)$$

If it exists, then $\text{Evol}(\gamma) := \eta$ is uniquely determined by (36) (see Lemma 5.12). If, moreover, smooth functions act smoothly on $AC_{\mathcal{E}}$, then we say that G is \mathcal{E} -regular if G is \mathcal{E} -semiregular and the map

$$\text{Evol}: \mathcal{E}([a, b], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([a, b], G)$$

is smooth.

Remark 5.17 Write $\gamma = [\zeta] \in \mathcal{E}([0, 1], \mathfrak{g})$ in the preceding definition. Then (36) is satisfied if and only if $\eta: [0, 1] \rightarrow G$ is a Carathéodory solution to the initial value problem

$$y' = f(t, y), \quad y(0) = e$$

with $f: [0, 1] \times G \rightarrow TG$, $f(t, y) := y\zeta(t)$ (using the left action $G \times TG \rightarrow TG$ given by $gv = T\lambda_g(v)$).

Remark 5.18 Lemma 4.8(c) shows that if a Lie group G over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is \mathcal{E} -regular and $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([0, 1], G)$ is smooth (resp., \mathbb{K} -analytic), then also

$$\text{evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow G, \quad \gamma \mapsto \text{Evol}(\gamma)(1)$$

is smooth (resp., \mathbb{K} -analytic). In contrast to the case of C^k -regularity, no exponential laws are available for spaces of measurable maps, whence we cannot deduce continuity or differentiability properties of Evol from such of evol in the current situation. In the case of measurable regularity properties, Evol (rather than evol) is the key object to work with, whose study provides the largest amount of information.

Definition 5.19 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex space) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act on $AC_{\mathcal{E}}$. Let G be a local Lie group modelled on a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space), with Lie algebra \mathfrak{g} and neutral element e . We say that G is *locally \mathcal{E} -semiregular* if there exists an open 0-neighbourhood

$$\Omega \subseteq \mathcal{E}([0, 1], \mathfrak{g})$$

such that, for each $\gamma \in \Omega$, there exists $\eta \in AC_{\mathcal{E}}([0, 1], G)$ such that

$$\delta^\ell(\eta) = \gamma \quad \text{and} \quad \eta(0) = e. \quad (37)$$

If it exists, then $\text{Evol}(\gamma) := \eta$ is uniquely determined by (37) (Lemma 5.12). If, moreover, G has a global chart and smooth functions act smoothly on $AC_{\mathcal{E}}$, then we say that G is *locally \mathcal{E} -regular* if G is locally \mathcal{E} -semiregular and Ω can be chosen such that

$$\text{Evol}: \Omega \rightarrow AC_{\mathcal{E}}([a, b], G)$$

is smooth.

Proposition 5.20 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex space) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let G be a \mathcal{E} -semiregular Lie group modelled on a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space) E , with Lie algebra \mathfrak{g} and neutral element e . Then the map

$$\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([0, 1], G)$$

is smooth if and only if Evol is smooth as a map

$$\mathcal{E}([0, 1], \mathfrak{g}) \rightarrow C([0, 1], G).$$

If G is a locally \mathcal{E} -semiregular local Lie group modelled on E admitting a global chart and $\Omega \subseteq \mathcal{E}([0, 1], \mathfrak{g})$ an open 0-neighbourhood on which Evol is defined, then

$$\text{Evol}: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], G)$$

is smooth if and only if Evol is smooth as a map $\Omega \rightarrow C([0, 1], G)$.

Proof. If Evol is smooth to $AC_{\mathcal{E}}([0, 1], G)$, then also to $C([0, 1], G)$, since the inclusion map $AC_{\mathcal{E}}([0, 1], G) \rightarrow C([0, 1], G)$ is smooth (cf. Remark 4.3).

Conversely, write Evol_C for Evol as a map to $C([0, 1], G)$ and assume that Evol_C is smooth. Let $U \subseteq G$ be a symmetric identity neighbourhood on which a chart $\phi: U \rightarrow V \subseteq E$ is defined. Let $\Omega \subseteq \mathcal{E}([0, 1], \mathfrak{g})$ be an open 0-neighbourhood such that $\text{Evol}_C(\Omega) \subseteq C([0, 1], U)$ and $\text{Evol}_C|_{\Omega}$ is smooth. Let us show that $\text{Evol}|_{\Omega}: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], G)$ is smooth, or equivalently, that

$$g := AC_{\mathcal{E}}([0, 1], \phi) \circ \text{Evol}|_{\Omega}: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], E)$$

is smooth. By Lemma 3.9 and [5, Lemma 10.1], the latter will hold if we can show that g is smooth as a map to $C([0, 1], E)$ (which is the case $C([0, 1], \phi) \circ \text{Evol}_C$ is smooth) and the map

$$h: \Omega \rightarrow \mathcal{E}([0, 1], E), \quad \gamma \mapsto (g(\gamma))'$$

is smooth. To see that h is smooth, consider U is a local Lie group with $D_U := \{(x, y) \in U \times U: xy \in U\}$ and make $V \subseteq E$ a local Lie group such that $\phi: U \rightarrow V$ is an isomorphism of local Lie groups. Then

$$\phi \circ \text{Evol}_G = \text{Evol}_V \circ \mathcal{E}([0, 1], L(\phi)).$$

Let $\mu: D_V \rightarrow V$ be the local group multiplication and

$$\nu: V \times L(V) \rightarrow TV, \quad (x, v) \mapsto T\mu(0_x, v).$$

Since $\eta' = \nu \circ (\eta, \delta^\ell \eta)$, we have

$$\begin{aligned} (g(\gamma))' &= (\text{Evol}_V(L(\phi) \circ \gamma))' = \nu \circ (\text{Evol}_V(L(\phi) \circ \gamma), L(\phi) \circ \gamma) \\ &= \nu \circ (\phi \circ \text{Evol}_C(\gamma), L(\phi) \circ \gamma) \\ &= \tilde{\nu}\left(C([0, 1], \phi)(\text{Evol}_C(\gamma)), \mathcal{E}([0, 1], L(\phi))(\gamma)\right) \end{aligned} \tag{38}$$

with $\widetilde{\nu}$ as in the pushforward axiom (P2). As the map $\mathcal{E}([0, 1], L(\phi))$ is continuous linear by the bifunctor axiom (B1) and the map $C([0, 1], \phi): C([0, 1], U) \rightarrow C([0, 1], V)$ is a C^∞ -diffeomorphism (being a chart of $C([0, 1], G)$), we deduce from (38) with the pushforward axiom (P2) that h is smooth. In the case of a local Lie group, this completes the proof. If G is a Lie group, let us show that Evol is smooth on an open neighbourhood of each $\gamma \in \mathcal{E}([0, 1], \mathfrak{g})$. Write $\eta := \text{Evol}(\gamma)$. Let $W \subseteq U$ be an open identity neighbourhood such that

$$W^{-1}WW \subseteq U.$$

Using a compactness argument, we can find $0 = t_0 < t_1 < \dots < t_n = 1$ with

$$\eta(t_{j-1})^{-1}\eta([t_{j-1}, t_j]) \subseteq W$$

for all $j \in \{1, \dots, n\}$. Then

$$Q := \{\zeta \in C([0, 1], G) : \eta^{-1}\zeta \in C([0, 1], W)\}$$

is an open neighbourhood of η in $C([0, 1], G)$ and hence

$$P := (\text{Evol}_C)^{-1}(Q)$$

is an open neighbourhood of γ in $\mathcal{E}([0, 1], \mathfrak{g})$. By Lemma 4.8 (b), $\text{Evol}|_P$ will be smooth if we can show that the map

$$P \rightarrow \prod_{j=1}^n AC_{\mathcal{E}}([t_{j-1}, t_j], G), \quad \tau \mapsto (\text{Evol}(\tau)|_{[t_{j-1}, t_j]})_{j \in \{1, \dots, n\}}$$

is smooth. The latter holds if, for each $j \in \{1, \dots, n\}$, the component

$$f_j: P \rightarrow AC_{\mathcal{E}}([t_{j-1}, t_j], G), \quad \tau \mapsto \text{Evol}(\tau)|_{[t_{j-1}, t_j]}$$

is smooth. It is well-known that the evaluation map $\varepsilon_{t_{j-1}}: C([a, b], G) \rightarrow G$, $\zeta \mapsto \zeta(t_{j-1})$ is smooth (cf. [20]). By Lemma 4.9, g_j is smooth if we can show that the map

$$P \rightarrow G, \quad \tau \mapsto \text{Evol}(\tau)(t_{j-1}) = \varepsilon_{t_{j-1}}(\text{Evol}_C(\tau))$$

is smooth (which is the case by smoothness of Evol_C and $\varepsilon_{t_{j-1}}$) and the map

$$g_j: P \rightarrow AC_{\mathcal{E}}([t_{j-1}, t_j], G)_*, \quad \tau \mapsto (f_j(\tau)(t_{j-1}))^{-1}f_j(\tau)$$

is smooth. Note that, if $\zeta \in Q$ and $t \in [t_{j-1}, t_j]$, then

$$\begin{aligned} g_j(\zeta)(t) &= \text{Evol}(\zeta)(t_{j-1})^{-1} \text{Evol}(\zeta)(t) \\ &= (\eta(t_{j-1})^{-1} \text{Evol}(\zeta)(t_{j-1}))^{-1} \eta(t_{j-1})^{-1} \eta(t) (\eta(t))^{-1} \text{Evol}(\zeta)(t) \\ &\in W^{-1} W W \subseteq U. \end{aligned}$$

We therefore only need to show that

$$AC_{\mathcal{E}}([t_{j-1}, t_j], \phi) \circ g_j : P \rightarrow AC_{\mathcal{E}}([t_{j-1}, t_j], V)$$

is smooth. As a map G_j to $C([t_{j-1}, t_j], E)$, this map is smooth by smoothness of Evol_C . Hence, by Lemma 3.9 and [5, Lemma 10.1], it only remains to show that the map

$$h_j : P \rightarrow \mathcal{E}([t_{j-1}, t_j], E), \quad \tau \mapsto (\phi \circ g_j(\tau))'$$

is smooth. But

$$h_j(\tau) = \tilde{\nu}(G_j(\tau), \mathcal{E}([t_{j-1}, t_j], L(\phi))(\tau))$$

with the smooth map

$$\tilde{\nu} : C([t_{j-1}, t_j], V) \times \mathcal{E}([t_{j-1}, t_j], E) \rightarrow \mathcal{E}([t_{j-1}, t_j], E), \quad \tilde{\nu}(\sigma, \tau) := \nu \circ (\sigma, \tau)$$

(as in the pushforward axiom (P2)), since

$$(\phi \circ g_j(\tau))' = \nu \circ (\phi \circ g_j(\tau), \delta^\ell(\phi \circ g_j(\tau)))$$

with $\delta^\ell(\phi \circ g_j(\tau)) = L(\phi) \circ \delta^\ell(\text{Evol}(\tau)(t_{j-1})^{-1} \text{Evol}(\tau)) = L(\phi) \circ \delta^\ell(\text{Evol}(\tau)) = L(\phi) \circ \tau$. Hence h_j is smooth and hence $\text{Evol}|_P$ is smooth. \square

We now obtain Theorem A as a special case of the following corollary:

Corollary 5.21 *Let G be a \mathcal{E} -semiregular Lie group modelled on a Fréchet space or a sequentially complete (FEP)-space. Let $p \in [1, \infty]$. Then we have the following implications:*

$$\begin{aligned} G \text{ is } L^p\text{-regular} &\Rightarrow G \text{ is } L^q\text{-regular for all } q \geq p; \\ G \text{ is } L^\infty\text{-regular} &\Rightarrow G \text{ is } L_{rc}^\infty\text{-regular} \Rightarrow G \text{ is } R\text{-regular}; \\ G \text{ is } R\text{-regular} &\Rightarrow G \text{ is } C^0\text{-regular}. \end{aligned}$$

Proof. Let $\mathfrak{g} := L(G)$. Since

$$C^0([0, 1], \mathcal{G}) \subseteq R([0, 1], \mathfrak{g}) \subseteq L_{rc}^\infty([0, 1], \mathfrak{g}) \subseteq L^q([0, 1], \mathfrak{g}) \subseteq L^p([0, 1], \mathfrak{g}) \quad (39)$$

with continuous linear inclusion maps, the \mathcal{E} -semiregularity with respect to a class \mathcal{E} of spaces further on the right in the chain (39) of inclusions implies \mathcal{F} -semiregularity with respect to each class \mathcal{F} of spaces further on the left. Let us write $\text{Evol}_\mathcal{E}$ and $\text{Evol}_\mathcal{F}$ for the respective evolution map and $j_{\mathcal{E}, \mathcal{F}}$ for the inclusion map $\mathcal{F}([0, 1], \mathfrak{g}) \rightarrow \mathcal{E}([0, 1], \mathfrak{g})$. We know that both \mathcal{E} and \mathcal{F} satisfy the locality axiom, the pushforward axioms, and that smooth functions act smoothly on $AC_\mathcal{E}$ and on $AC_\mathcal{F}$. Now $\text{Evol}_\mathcal{F} = \text{Evol}_\mathcal{E} \circ j_{\mathcal{E}, \mathcal{F}}$ as mappings to $C([0, 1], G)$. If G is \mathcal{E} -regular, then $\text{Evol}_\mathcal{E}$ is smooth (see Proposition 5.20) and hence also $\text{Evol}_\mathcal{F} = \text{Evol}_\mathcal{E} \circ j_{\mathcal{E}, \mathcal{F}}$ is smooth. Thus, again by Proposition 5.20, G is \mathcal{F} -regular. \square

A similar argument shows:

Corollary 5.22 *Let G be a Lie group modelled on an integral complete locally convex space. If G is L_{rc}^∞ -regular, then G is also R -regular.* \square

Remark 5.23 Analogous implications are available for local Lie groups.

Definition 5.24 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces). Let E be a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space) and $[\gamma] \in \mathcal{E}([0, 1], E)$. For $n \in \mathbb{N}$ and $k \in \{0, 1, \dots, n-1\}$, define

$$\gamma_{n,k}: [0, 1] \rightarrow E, \quad \gamma_{n,k}(t) := \frac{1}{n} \gamma((k+t)/n).$$

Then $[\gamma_{n,k}] \in \mathcal{E}([0, 1], E)$ for all $n \in \mathbb{N}$ and $k \in \{0, 1, \dots, n-1\}$ (by axiom (B2)). We say that \mathcal{E} has the *subdivision property* if the following holds for each Fréchet space (resp., sequentially complete (FEP)-space, resp., each integral complete locally convex space) E :

For each $\gamma \in \mathcal{E}([0, 1], E)$ and continuous seminorm q on $\mathcal{E}([0, 1], E)$, we have that

$$\sup_{k \in \{0, 1, \dots, n-1\}} q(\gamma_{n,k}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proposition 5.25 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. If, moreover, \mathcal{E} has the subdivision property, then the following conditions are equivalent for each Lie group G modelled on a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space):*

- (a) G is \mathcal{E} -regular;
- (b) G is locally \mathcal{E} -regular.

Also the following conditions are equivalent if \mathcal{E} has the subdivision property:

- (c) G is \mathcal{E} -semiregular;
- (d) G is locally \mathcal{E} -semiregular.

Proof. Let $\mathfrak{g} := L(G)$. If G is \mathcal{E} -semiregular (resp., \mathcal{E} -regular), then trivially G is also locally \mathcal{E} -semiregular (resp., locally \mathcal{E} -regular).

Now assume that G is locally \mathcal{E} -regular (resp., \mathcal{E} -semiregular). Thus, there exists an open 0-neighbourhood $\Omega \subseteq \mathcal{E}([0, 1], \mathfrak{g})$ such that $\text{Evol}(\gamma) \in AC_{\mathcal{E}}([0, 1], G)$ exists for each $\gamma \in \Omega$ (resp., moreover $\text{Evol}: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], G)$ is smooth). After shrinking Ω , we may assume that $\Omega = B_1^q(0)$ for a continuous seminorm $q: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow [0, \infty[$. Now let $\gamma \in \mathcal{E}([0, 1], \mathfrak{g})$. By the subdivision property, we find $n \in \mathbb{N}$ such that

$$\gamma_{n,k} \in \Omega \quad \text{for all } k \in \{0, 1, \dots, n-1\}.$$

By Axiom (B2), the linear map

$$\alpha_k: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow \mathcal{E}([0, 1], \mathfrak{g}), \quad \eta \mapsto \eta_{n,k}$$

is continuous for all $k \in \{0, 1, \dots, n-1\}$. Hence, we find an open neighbourhood $W \subseteq \mathcal{E}([0, 1], \mathfrak{g})$ of γ such that

$$\eta_{n,k} \in \Omega \quad \text{for all } \eta \in W \text{ and all } k \in \{0, 1, \dots, n-1\}.$$

For $\eta \in W$, define $\text{Evol}(\eta): [0, 1] \rightarrow G$ via

$$\text{Evol}(\eta)(t) := \text{Evol}(\eta_{n,0})(nt) \quad \text{if } t \in [0, \frac{1}{n}]$$

and

$$\text{Evol}(\eta)(t) := \text{Evol}(\eta_{n,0})(1) \cdots \text{Evol}(\eta_{n,k-1})(1) \text{Evol}(\eta_{n,k})(nt - k)$$

if $t \in [\frac{k}{n}, \frac{k+1}{n}]$ with $k \in \{1, \dots, n-1\}$. The map $\text{Evol}(\eta)$ is continuous and $\text{Evol}(\eta)|_{[k/n, (k+1)/n]}$ is in $AC_{\mathcal{E}}([k/n, (k+1)/n], G)$ (since $\text{Evol}(\eta_{n,k}) \in AC_{\mathcal{E}}([0, 1], G)$), as a consequence of axiom (B2). So $\text{Evol}(\eta) \in AC_{\mathcal{E}}([0, 1], G)$. Since $\text{Evol}(\eta)(0) = e$ and $\text{Evol}(\eta)$ has left logarithmic derivative η by construction, indeed $\text{Evol}(\eta)$ is a left evolution for η . Notably, $\text{Evol}(\gamma)$ is a left evolution for γ and hence G is \mathcal{E} -semiregular. If $\text{Evol}: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], G)$ is smooth, then also

$$\text{Evol}: W \rightarrow AC_{\mathcal{E}}([0, 1], G), \quad \eta \mapsto \text{Evol}(\eta)$$

as just defined is smooth. To see this, we re-use that the map

$$AC_{\mathcal{E}}([0, 1], G) \rightarrow \prod_{k=0}^{n-1} AC_{\mathcal{E}}([k/n, (k+1)/n], G), \quad \zeta \mapsto (\zeta|_{[k/n, (k+1)/n]})_{k=0,1,\dots,n-1}$$

is an isomorphism of Lie groups onto a Lie subgroup. We therefore only need to show that

$$W \rightarrow AC_{\mathcal{E}}([k/n, (k+1)/n], G), \quad \eta \mapsto \text{Evol}(\eta)|_{[k/n, (k+1)/n]} \quad (40)$$

is smooth. For fixed k , the Lie group $AC_{\mathcal{E}}([k/n, (k+1)/n], G)$ is isomorphic to $AC_{\mathcal{E}}([0, 1], G)$ via $\zeta \mapsto (t \mapsto \zeta((k+t)/n))$ and the composition of this isomorphism and Evol_W is the map

$$W \rightarrow AC_{\mathcal{E}}([0, 1], G), \quad \eta \mapsto \text{evol}(\eta_{n,0}) \cdots \text{evol}(\eta_{n,k-1}) \text{Evol}(\eta_{n,k}).$$

This map is smooth as it is the product of compositions of $\text{Evol}: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], G)$ (or $\text{evol}: \Omega \rightarrow G \subseteq AC_{\mathcal{E}}([0, 1], G)$) and the continuous linear maps $\eta \mapsto \eta_{n,j}$ for $j \in \{0, \dots, k\}$. We deduce that the map in (40) (and hence also $\text{Evol}: W \rightarrow AC_{\mathcal{E}}([0, 1], G)$) is smooth. \square

Lemma 5.26 *L^p as a bifunctor on Fréchet spaces (or sequentially complete (FEP)-spaces) has the subdivision property, for all $p \in [1, \infty]$. Moreover, L_{rc}^∞ and R have the subdivision property as bifunctors on integral complete locally convex spaces.*

Proof. In the case when \mathcal{E} is L^∞ , L_{rc}^∞ or R , we have

$$\begin{aligned}\|\gamma_{n,k}\|_{L^\infty,q} &= \operatorname{ess\,sup}_{t \in [0,1]} \frac{1}{n} q(\gamma((k+t)/n)) \\ &\leq \frac{1}{n} \operatorname{ess\,sup}_{t \in [0,1]} \frac{1}{n} q(\gamma(t)) = \frac{1}{n} \|\gamma\|_{L^\infty,q}\end{aligned}$$

for all $[\gamma] \in \mathcal{E}([0,1], E)$, $q \in P(E)$, $n \in \mathbb{N}$, $q \in P(E)$ and $k \in \{0, 1, \dots, n-1\}$. Hence

$$\max_{k \in \{0,1,\dots,n-1\}} \|\gamma_{n,k}\|_{L^\infty,q} \leq \frac{1}{n} \|\gamma\|_{L^\infty,q} \rightarrow 0$$

as $n \rightarrow \infty$, showing that the subdivision property is satisfied.

Now assume that $\mathcal{E} = L^p$ with $p \in [1, \infty[$. Let $\gamma \in \mathcal{L}^p([0,1], E)$. Substituting $s = (k+t)/n$, we see that

$$\begin{aligned}\|\gamma_{n,k}\|_{\mathcal{L}^p,q} &= \sqrt[p]{\int_0^1 q(\gamma((k+t)/n))^p \frac{dt}{n^p}} \\ &\leq \sqrt[p]{\int_0^1 q(\gamma((k+t)/n))^p \frac{dt}{n}} = \sqrt[p]{\int_{k/n}^{(k+1)/n} q(\gamma(s)) ds} \leq \|\gamma\|_{\mathcal{L}^p,q}\end{aligned}$$

for each continuous seminorm q on E . Thus $\|\gamma_{n,k}\|_{\mathcal{L}^p,q} \leq \|\gamma\|_{\mathcal{L}^p,q}$ for all $n \in \mathbb{N}$ and $k \in \{0, 1, \dots, n-1\}$.

Let $\varepsilon > 0$. For $m \in \mathbb{N}$, define

$$A_m := \{t \in [0,1] : q(\gamma(t)) > m\}.$$

Then each A_n is a Borel set in $[0,1]$, we have $A_1 \subseteq A_2 \supseteq \dots$, and $\bigcap_{m \in \mathbb{N}} A_m = \emptyset$. Thus $\mathbf{1}_{A_m} \rightarrow 0$ holds pointwise for the characteristic functions of the set A_m . Hence, by dominated convergence,

$$\int_{A_m} q(\gamma(t))^p dt = \int_0^1 q(\gamma(t))^p \mathbf{1}_{A_m} dt \rightarrow 0$$

as $m \rightarrow \infty$. We therefore find $m \in \mathbb{N}$ such that

$$\int_{A_m} q(\gamma(t))^p dt \leq \varepsilon^p/2$$

Choose $n_0 \in \mathbb{N}$ so large that $m/n_0 \leq \varepsilon/\sqrt[p]{2}$. Given $n \geq n_0$, define

$$A_{n,k} := \{t \in [0,1] : (k+t)/n \in A_m\}$$

for $k \in \{0, 1, \dots, n-1\}$. Note that, if $t \in [0, 1] \setminus A_{n,k}$, then $(k+1)/n \in [0, 1] \setminus A_m$ and hence

$$q(\gamma_{n,k}(t))^p = \left(\frac{1}{n} q(\gamma((k+t)/n)) \right)^p \leq (m/n)^p \leq (m/n_0)^p \leq \varepsilon^p/2.$$

Moreover, substituting $s = (k+t)/n$,

$$\int_{A_{n,k}} q(\gamma_{n,k}(t))^p dt = \frac{n}{n^p} \int_{A_m \cap [k/n, (k+1)/n]} q(\gamma(s))^p ds \leq \varepsilon^p/2.$$

Thus

$$\begin{aligned} \|\gamma_{n,k}\|_{\mathcal{L}^p,q} &= \sqrt[p]{\int_0^1 q(\gamma_{n,k}(t)) dt} \\ &= \sqrt[p]{\int_{A_{n,k}} q(\gamma_{n,k}(t))^p dt + \int_{[0,1] \setminus A_{n,k}} \underbrace{q(\gamma_{n,k}(t))^p}_{\leq \varepsilon^p/2} dt} \\ &\leq \sqrt[p]{\varepsilon^p/2 + \varepsilon^p/2} = \varepsilon \end{aligned}$$

and thus

$$\max_{k \in \{0, 1, \dots, n-1\}} \|\gamma_{n,k}\|_{\mathcal{L}^p,q} \leq \varepsilon$$

for all $n \geq n_0$. Therefore, the subdivision property is satisfied. \square

Proposition 5.27 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex space) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act on $AC_{\mathcal{E}}$. Let G be a Lie group modelled on a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space) E , with Lie algebra \mathfrak{g} and neutral element e . Let $H \subseteq G$ be a subgroup which is a submanifold of G , modelled on a closed vector subspace $F \subseteq E$. Assume that*

$$H = \{x \in G : (\forall j \in J) \alpha_j(x) = \beta_j(x)\} \quad (41)$$

for Lie groups H_j modelled on Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) and smooth homomorphisms $\alpha_j, \beta_j : G \rightarrow H_j$. Also, assume that $F \subseteq E$ is complemented or that \mathcal{E} satisfies the embedding axiom. Then the following holds:

- (a) If G is \mathcal{E} -semiregular, then also H is \mathcal{E} -semiregular.
- (b) If smooth functions act smoothly on $AC_{\mathcal{E}}$ and G is \mathcal{E} -regular, then also H is \mathcal{E} -regular.

Proof. (a) Let $\mathfrak{g} := L(G)$ and $\mathfrak{h} := L(H) \subseteq \mathfrak{g}$. If $\gamma \in \mathcal{E}([0, 1], \mathfrak{h})$, then $\gamma \in \mathcal{E}([0, 1], \mathfrak{g})$ and thus $\eta := \text{Evol}(\gamma) \in AC_{\mathcal{E}}([0, 1], G)$ exists. For each $j \in J$,

$$\alpha_j \circ \eta \quad \text{and} \quad \beta_j \circ \eta$$

are elements of $AC_{\mathcal{E}}([0, 1], H_j)$. If $\lambda: H \rightarrow G$ is the inclusion map, then

$$\alpha_j \circ \lambda = \beta_j \circ \lambda,$$

entailing that $L(\alpha_j) \circ L(\lambda) = L(\beta_j) \circ L(\lambda)$ and hence

$$L(\alpha_j)|_{\mathfrak{h}} = L(\beta_j)|_{\mathfrak{h}}.$$

Hence

$$\begin{aligned} \delta^\ell(\alpha_j \circ \eta) &= L(\alpha_j) \circ \delta^\ell(\eta) = L(\alpha_j) \circ \gamma \\ &= L(\alpha_j)|_{\mathfrak{h}} \circ \gamma = L(\beta_j)|_{\mathfrak{h}} \circ \gamma = \delta^\ell(\beta_j \circ \eta) \end{aligned}$$

and thus $\alpha_j \circ \eta = \beta_j \circ \eta$. As j was arbitrary, we deduce that $\eta([0, 1]) \subseteq H$. Since H is a submanifold of G and $\eta \in AC_{\mathcal{E}}([0, 1], G)$ with $\eta([0, 1]) \subseteq H$, we obtain $\eta \in AC_{\mathcal{E}}([0, 1], H)$ with Lemma 4.12, as the assume that F is complemented in E or \mathcal{E} satisfies the embedding axiom. By construction, $\delta^\ell \eta = \gamma$ and thus $\eta = \text{Evol}(\gamma) \in AC_{\mathcal{E}}([0, 1], H)$.

(b) By (a), H is \mathcal{E} -semiregular and

$$\lambda \circ \text{Evol}_H = \text{Evol}_G|_{\mathcal{E}([0, 1], \mathfrak{h})}, \tag{42}$$

if $\lambda: H \rightarrow G$ is the inclusion map and $\text{Evol}_G: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([0, 1], G)$ as well as $\text{Evol}_H: \mathcal{E}([0, 1], \mathfrak{h}) \rightarrow AC_{\mathcal{E}}([0, 1], H)$ are the respective evolution maps. Since $\text{Evol}_G|_{\mathcal{E}([0, 1], \mathfrak{h})}$ is smooth and $AC_{\mathcal{E}}([0, 1], H) \subseteq AC_{\mathcal{E}}([0, 1], G)$ is a submanifold (Lemma 4.13), we deduce from (42) that Evol_H is smooth. \square

Remark 5.28 Assume that G is replaced with a local Lie group in the preceding proposition, with domain D_G for the multiplication. Assume that H_j is a submanifold of G and that α_j, β_j are smooth homomorphisms of local

groups from G to local Lie groups H_j such that (41) holds. Also, assume that G admits a global chart which restricts to a global chart for H . If the modelling space of H is complemented in that of G or \mathcal{E} satisfies the embedding axiom, then H with $D_H := \{(x, y) \in D_G \cap (H \times H) : xy \in H\}$ is a local Lie group and we have:

- (a) If G is locally \mathcal{E} -semiregular, then also H is locally \mathcal{E} -semiregular.
- (b) If smooth functions act smoothly on $AC_{\mathcal{E}}$ and G is locally \mathcal{E} -regular, then also H is locally \mathcal{E} -regular.

The proof follows the same lines.

Lemma 5.29 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex space) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let G be a Lie group (or local Lie group with global chart) modelled on a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space), with Lie algebra \mathfrak{g} and neutral element e . Then the map*

$$\delta^\ell : AC_{\mathcal{E}}([0, 1], G) \rightarrow \mathcal{E}([0, 1], \mathfrak{g}), \quad \eta \mapsto \delta^\ell(\eta)$$

is smooth and

$$d(\delta^\ell)(\eta) = \eta' \quad \text{for all } \eta \in AC_{\mathcal{E}}([0, 1], \mathfrak{g}) \quad (43)$$

if we identify the Lie algebra $T_e AC_{\mathcal{E}}([0, 1], G)$ with $AC_{\mathcal{E}}([0, 1], \mathfrak{g})$ by means of the isomorphism $dAC_{\mathcal{E}}([0, 1], \phi)|_{T_e AC_{\mathcal{E}}([0, 1], G)}$, for $U \subseteq G$ an open symmetric identity neighbourhood and $\phi : U \rightarrow V \subseteq \mathfrak{g}$ a C^∞ -diffeomorphism such that $d\phi|_{\mathfrak{g}} = \text{id}_{\mathfrak{g}}$.

Proof. Step 1. Assume that we can show that $\delta^\ell|_W$ is smooth for some identity neighbourhood $W \subseteq AC_{\mathcal{E}}([0, 1], G)$. Then also $\delta^\ell|_{W\eta}$ is smooth, for each $\eta \in AC_{\mathcal{E}}([0, 1], G)$ (as we now verify) and thus δ^ℓ is smooth. In fact, for each $\zeta \in W\eta$ we have

$$\delta^\ell(\zeta) = \delta^\ell((\zeta\eta^{-1})\eta) = \text{Ad}(\eta^{-1})(\delta^\ell|_W(\zeta\eta^{-1})) + \delta^\ell(\eta).$$

As the second summand is constant (i.e., independent of ζ) and the map $W\eta \rightarrow W$, $\zeta \mapsto \zeta\eta^{-1}$ is smooth, it only remains to show that the map

$$h : \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow \mathcal{E}([0, 1], \mathfrak{g}), \quad [\gamma] \mapsto [t \mapsto \text{Ad}(\eta(t)^{-1})(\gamma(t))]$$

is smooth. Now

$$f: G \times \mathfrak{g} \rightarrow \mathfrak{g}, \quad f(x, y) := \text{Ad}_x(y)$$

is a smooth map which is linear in its second argument. Since $h = \tilde{f}(\eta^{-1}, \cdot)$ with

$$\tilde{f}: C([0, 1], G) \times \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow \mathcal{E}([0, 1], \mathfrak{g}), \quad (\tau, [\sigma]) \mapsto [f \circ (\tau, \sigma)],$$

we deduce with Lemma 5.10 that h is continuous linear and hence smooth.

Step 2. Let $\phi: U \rightarrow V \subseteq E$ be a chart of G with $\phi(e) = 0$, defined on an open symmetric identity neighbourhood U . Then U is a local Lie group with $D_U := \{(x, y) \in U \times U : xy \in U\}$. We give V the local Lie group structure which makes $\phi: U \rightarrow V$ an isomorphism of local Lie groups. Consider the smooth map

$$\omega: TV \rightarrow L(V), \quad \omega(v) = T\mu(0_{\pi_{TV}(x)^{-1}}, v).$$

groups. If we identify TV with $V \times E$ and $L(V) = T_0V = \{0\} \times E$ with E (via $(0, y) \mapsto y$), then ω becomes the map

$$\omega: V \times E \rightarrow E, \quad \omega(x, y) = d\mu(x^{-1}, x; 0, y).$$

For $\eta \in AC_{\mathcal{E}}([0, 1], V)$ with $\eta' = [\gamma]$, the left logarithmic derivative is

$$\delta_V^\ell(\eta) = [\omega \circ (\eta, \gamma)].$$

Since ω is linear in its second argument and smooth, the map

$$\tilde{\omega}: C([0, 1], V) \times \mathcal{E}([0, 1], E) \rightarrow \mathcal{E}([0, 1], E), \quad \tilde{\omega}(\tau, [\sigma]) := [\omega \circ (\tau, \sigma)]$$

is smooth, by the pushforward axiom (P2). Hence

$$\delta_V^\ell: AC_{\mathcal{E}}([0, 1], V) \rightarrow \mathcal{E}([0, 1], E), \quad \eta \mapsto \tilde{\omega}(\eta, \eta')$$

is smooth. Now, for $s \neq 0$ close to 0:

$$\frac{\delta_V^\ell(s\eta) - \overbrace{\delta_V^\ell(0)}^{=0}}{s} = \tilde{\omega}(s\eta, [\gamma]) \rightarrow \tilde{\omega}(0, [\gamma])$$

as $s \rightarrow 0$. Since $d\mu(0, 0, 0, y) = y$ for all $y \in E$, we deduce that

$$d(\delta^\ell)(0, \eta) = \tilde{\omega}(0, [\gamma]) = [t \mapsto d\mu(0, 0, 0, \gamma(t))] = [\gamma] = \eta'.$$

To complete the proof, let us write δ_U^ℓ for the restriction of the map $\delta^\ell: AC_\mathcal{E}([0, 1], G) \rightarrow \mathcal{E}([0, 1], \mathfrak{g})$ to $AC_\mathcal{E}([0, 1], V)$. Then

$$L(\phi) \circ \delta_U^\ell(\eta) = \delta_V^\ell(\phi \circ \eta)$$

for all $\eta \in AC_\mathcal{E}([0, 1], U)$ and thus

$$\mathcal{E}([0, 1], L(\phi)) \circ \delta_U^\ell = \delta_V^\ell \circ AC_\mathcal{E}([0, 1], \phi).$$

Taking the differential at $\eta = e$, we obtain

$$\mathcal{E}([0, 1], L(\phi)) \circ d\delta_U^\ell = \delta_V^\ell \circ T_e AC_\mathcal{E}([0, 1], \phi)$$

on $T_e AC_\mathcal{E}([0, 1], G)$. Composing with $T_e AC_\mathcal{E}([0, 1], \phi)^{-1}$ on the right and with $\mathcal{E}([0, 1], L(\phi))^{-1}$ on the left, (43) follows. \square

Remark 5.30 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act smoothly on $AC_\mathcal{E}$. Let G be a Lie group (or local Lie group with global chart) modelled on such a space, with Lie algebra \mathfrak{g} and neutral element e . Since $\delta^\ell(\text{Evol}(\gamma)) = \gamma$, Lemma 5.29 and the Chain Rule entail:

- (a) If G is \mathcal{E} -regular (resp., locally \mathcal{E} regular), then

$$T_0 \text{Evol}(\gamma)(t) = \int_0^t \gamma(s) ds \quad \text{for all } t \in [0, 1],$$

for all $\gamma \in \mathcal{E}([0, 1], \mathfrak{g}) \cong \{0\} \times \mathcal{E}([0, 1], \mathfrak{g}) = T_0 \mathcal{E}([0, 1], \mathfrak{g})$, identifying $T_e AC_\mathcal{E}([0, 1], G)$ with $AC_\mathcal{E}([0, 1], \mathfrak{g})$ as in Lemma 5.29. More generally:

- (b) If $F \subseteq \mathcal{E}([0, 1], E)$ is a vector subspace and $\Omega \subseteq F$ an open 0-neighbourhood for some locally convex vector topology on F such that $\text{Evol}(\gamma) \in AC_\mathcal{E}([0, 1], G)$ exists for all $\gamma \in \Omega$ and

$$\text{Evol}_\Omega: \Omega \rightarrow AC_\mathcal{E}([0, 1], G), \quad \gamma \mapsto \text{Evol}(\gamma)$$

is C^1 , then $T_0(\text{Evol}_\Omega)(\gamma)(t) = \int_0^t \gamma(s) ds$ for all $t \in [0, 1]$ and $\gamma \in F \cong \{0\} \times F = T_0 F$. More generally:

- (c) If $\gamma \in \mathcal{E}([0, 1], \mathfrak{g})$ such that $\text{Evol}(r\gamma) \in AC_{\mathcal{E}}([0, 1], G)$ exists for all r in a non-degenerate interval $J \subseteq \mathbb{R}$ with $0 \in J$ and

$$\left. \frac{d}{dr} \right|_{r=0} \text{Evol}(r\gamma)$$

exists,²⁵ then

$$\left(\left. \frac{d}{dr} \right|_{r=0} \text{Evol}(r\gamma) \right) (t) = \int_0^t \gamma(s) ds,$$

identifying $T_e AC_{\mathcal{E}}([0, 1], G)$ with $AC_{\mathcal{E}}([0, 1], \mathfrak{g})$ as above.

[Proof: Abbreviate $\eta := \left. \frac{d}{dt} \right|_{r=0} \text{Evol}(r\gamma)$. Since $\delta^\ell(\text{Evol}(r\gamma)) = r\gamma$ Lemma 1.57 and (43) entail that

$$\gamma = \left. \frac{d}{dr} \right|_{r=0} \delta^\ell(\text{Evol}(r\gamma)) = d(\delta^\ell)(\eta) = \eta'$$

and thus $\eta(t) = \eta(0) + \int_0^t \gamma(s) ds$. Since $\text{Evol}(\gamma) \in AC_{\mathcal{E}}([0, 1], \mathfrak{g})_*$, we have $\eta \in T_0 AC_{\mathcal{E}}([0, 1], \mathfrak{g})_* = \{\zeta \in AC_{\mathcal{E}}([0, 1], \mathfrak{g}) : \zeta(0) = 0\}$ (using the above identification). Thus $\eta(t) = \int_0^t \gamma(s) ds$.]

5.31 Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $\mathbb{K} = \mathbb{R}$, let $r \in \mathbb{N} \cup \{\infty, \omega\}$; if $\mathbb{K} = \mathbb{C}$, let $r = \omega$. Following [36], a $C_{\mathbb{K}}^r$ -map $f: M \rightarrow N$ between $C_{\mathbb{K}}^r$ -manifolds modelled on locally convex topological \mathbb{K} -vector spaces E and F is called a $C_{\mathbb{K}}^r$ -*submersion* if, for each $x \in M$, there exists a chart $\phi: U_\phi \rightarrow V_\phi \subseteq E$ of M with $x \in U_\phi$ and a chart $\psi: U_\psi \rightarrow V_\psi \subseteq F$ of N such that $f(U_\phi) \subseteq U_\psi$ and

$$\psi \circ f \circ \phi^{-1} = \pi|_{V_\phi}$$

for a continuous linear map $\pi: E \rightarrow F$ which admits a continuous linear right inverse $\sigma: F \rightarrow E$ (i.e., $\pi \circ \sigma = \text{id}_F$).

5.32 Assume $r \in \{\infty, \omega\}$ if $\mathbb{K} = \mathbb{R}$ and $r = \omega$ if $\mathbb{K} = \mathbb{C}$. It is known that a surjective $C_{\mathbb{K}}^r$ -homomorphism $q: G \rightarrow Q$ between $C_{\mathbb{K}}^r$ -Lie groups is a $C_{\mathbb{K}}^r$ -submersion if and only if $N := \ker(q)$ is a $C_{\mathbb{K}}^r$ -Lie subgroup of G and $q: G \rightarrow Q$ is an N -principal bundle of class $C_{\mathbb{K}}^r$, i.e., it admits local $C_{\mathbb{K}}^r$ -sections (cf. [36]).

²⁵We require that the limit exists in some chart for $AC_{\mathcal{E}}([0, 1], G)$ around e (and hence in every chart around e , by Lemma 1.57).

Our next proposition subsumes Theorem G.

Proposition 5.33 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex space) which satisfies the locality axiom, the pushforward axioms, the subdivision property, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let G be a Lie group. Assume that $N \subseteq G$ is a normal Lie subgroup such that G/N can be given a smooth Lie group structure making*

$$q: G \rightarrow G/N, \quad x \mapsto xN$$

a smooth submersion such that both N and G/N are modelled on a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space) and both N and G/N are \mathcal{E} -regular. Then also G is modelled on a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space) and G is \mathcal{E} -regular.

Proof. Let $\mathfrak{g} := L(G)$, $\mathfrak{n} := L(N)$ and $\mathfrak{q} := L(Q)$; thus $L(q): \mathfrak{g} \rightarrow \mathfrak{q}$ is a continuous linear map with kernel \mathfrak{n} admitting a continuous linear right inverse, entailing that

$$\mathfrak{g} \cong \mathfrak{n} \oplus \mathfrak{q}$$

is a Fréchet space (resp., a sequentially complete (FEP)-space, resp., an integral complete locally convex space). Let

$$\phi: U \rightarrow V$$

be a chart for G , defined on an open symmetric identity neighbourhood $U \subseteq G$. Let $W \subseteq Q$ be an open symmetric identity neighbourhood on which a smooth section $\sigma: W \rightarrow G$ is defined (thus $q \circ \sigma = \text{id}_W$). After shrinking W , we may assume that $\sigma(W) \subseteq U$ and that there is a chart $\psi: W \rightarrow W_1$ for Q defined on W . By hypothesis, we have smooth evolution maps

$$\text{Evol}_H: \mathcal{E}([0, 1], \mathfrak{h}) \rightarrow AC_{\mathcal{E}}([0, 1], H)$$

and $\text{Evol}_Q: \mathcal{E}([0, 1], \mathfrak{q}) \rightarrow AC_{\mathcal{E}}([0, 1], Q)$. Since Evol_Q is continuous, there is an open 0-neighbourhood $P \subseteq \mathcal{E}([0, 1], \mathfrak{q})$ such that

$$\text{Evol}_Q(\gamma) \in AC_{\mathcal{E}}([0, 1], W) \quad \text{for all } \gamma \in P.$$

Then

$$\Omega := \{\gamma \in \mathcal{E}([0, 1], \mathfrak{g}) : L(q) \circ \gamma \in P\}$$

is an open 0-neighbourhood in $\mathcal{E}([0, 1], \mathfrak{g})$ such that

$$\text{Evol}_Q(L(q) \circ \gamma) \in AC_{\mathcal{E}}([0, 1], W)$$

for all $\gamma \in \Omega$. Then

$$\zeta := \sigma \circ \text{Evol}_Q(L(q) \circ \gamma) \in AC_{\mathcal{E}}([0, 1], U) \subseteq AC_{\mathcal{E}}([0, 1], G)$$

and

$$L(q) \circ \delta^\ell(\zeta) = \delta^\ell(q \circ \zeta) = \delta^\ell \text{Evol}_Q(L(q) \circ \gamma) = L(q) \circ \gamma,$$

entailing that the function

$$\gamma - \delta^\ell(\zeta)$$

takes its values in $\ker L(q) = \mathfrak{n}$. As \mathfrak{n} is complemented in \mathfrak{g} and $\gamma - \delta^\ell(\zeta) \in \mathcal{E}([0, 1], \mathfrak{g})$, we obtain

$$\tau := \gamma - \delta^\ell(\zeta) \in \mathcal{E}([0, 1], \mathfrak{h}).$$

For $\theta \in AC_{\mathcal{E}}([0, 1], \mathfrak{n})$, we have

$$\begin{aligned} \delta^\ell(\theta\zeta) = \gamma &\Leftrightarrow \text{Ad}(\zeta)^{-1}(\delta^\ell(\theta)) + \delta^\ell(\zeta) = \gamma \\ &\Leftrightarrow \delta^\ell\theta = \text{Ad}(\zeta)(\tau). \end{aligned} \tag{44}$$

Note that $\text{Ad}(g).\mathfrak{n} \subseteq \mathfrak{n}$ for each $g \in G$ since N is a normal Lie subgroup of G (see [38]) and that

$$f: G \times \mathfrak{n} \rightarrow \mathfrak{n}, \quad f(g, v) := \text{Ad}_g(v)$$

is a smooth map which is linear in the second argument. By the pushforward axiom (P2), the associated map

$$\tilde{f}: C([0, 1], U) \times \mathcal{E}([0, 1], \mathfrak{n}) \rightarrow \mathcal{E}([0, 1], \mathfrak{n}), \quad \tilde{f}(\alpha, [\beta]) := [f \circ (\alpha, \beta)]$$

is smooth. In particular, we have

$$\text{Ad}(\zeta)(\tau) = \tilde{f}(\zeta, \tau) \in \mathcal{E}([0, 1], \mathfrak{n}),$$

enabling us to define

$$\theta := \text{Evol}_N(\text{Ad}(\zeta)(\tau)).$$

Then $\theta\zeta = \text{Evol}_G(\gamma)$, by (44), and thus G is locally \mathcal{E} -semiregular. Note that the map

$$g: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], G), \quad \gamma \mapsto \zeta = \sigma \circ \text{Evol}_Q(\mathcal{E}([0, 1], L(q))(\gamma))$$

is smooth. We use here the hypothesis that smooth functions act smoothly on $AC_{\mathcal{E}}$; hence

$$AC_{\mathcal{E}}([0, 1], \phi: \sigma \circ \psi^{-1}): AC_{\mathcal{E}}([0, 1], W_1) \rightarrow AC_{\mathcal{E}}([0, 1], V)$$

is smooth and hence also the map $AC_{\mathcal{E}}([0, 1], W) \rightarrow AC_{\mathcal{E}}([0, 1], U)$, $\alpha \mapsto \sigma \circ \alpha$, which is the following composition of smooth maps:

$$AC_{\mathcal{E}}([0, 1], \phi)^{-1} \circ AC_{\mathcal{E}}([0, 1], \phi: \sigma \circ \psi^{-1}) \circ AC_{\mathcal{E}}([0, 1], \psi).$$

The map

$$\delta^\ell: AC_{\mathcal{E}}([0, 1], U) \rightarrow \mathcal{E}([0, 1], \mathfrak{g})$$

is smooth (see Lemma 5.29), entailing that

$$h: \Omega \rightarrow \mathcal{E}([0, 1], \mathfrak{g}), \quad \gamma \mapsto \tau := \gamma - \delta^\ell(\zeta) = \gamma - \delta^\ell(g(\gamma))$$

is smooth. As \mathfrak{n} is complemented in \mathfrak{n} , we may consider g as a map to $\mathcal{E}([0, 1], \mathfrak{n})$. Now the formula

$$\text{Evol}(\gamma) = \theta\zeta = \text{Evol}_N(\tilde{f}(g(\gamma), h(\gamma)))g(\gamma)$$

shows that $\text{Evol}: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], G)$ is smooth. Hence G is locally \mathcal{E} -regular. As we assume that \mathcal{E} satisfies the subdivision property, we deduce with Proposition 5.25 that G is \mathcal{E} -regular. \square

As in the study of C^k -semiregularity [34], it is very useful for refined results to have a group structure on $\mathcal{E}([0, 1], \mathfrak{g})$ available if G is \mathcal{E} -semiregular.

Definition 5.34 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., sequentially complete (FEP)-spaces, resp., integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let G be a Lie group modelled on such a space, with Lie algebra \mathfrak{g} and neutral element e . If G is \mathcal{E} -semiregular, then the map

$$\delta^\ell: AC_{\mathcal{E}}([0, 1], G)_* \rightarrow \mathcal{E}([0, 1], \mathfrak{g})$$

is a bijection such that $\delta^\ell(e) = 0$. We can therefore make $\mathcal{E}([0, 1], \mathfrak{g})$ a group with group multiplication \odot and neutral element 0 in such a way that $\delta^\ell: AC_\mathcal{E}([0, 1], G)_* \rightarrow \mathcal{E}([0, 1], \mathfrak{g})$ becomes an isomorphism of groups. Then also

$$\text{Evol} = (\delta^\ell)^{-1}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow AC_\mathcal{E}([0, 1], G)_*$$

is an isomorphism of groups. We have

$$[\gamma_1] \odot [\gamma_2] = [\text{Ad}(\text{Evol}([\gamma_2]))^{-1} \gamma_1 + \gamma_2]$$

$$[\gamma_1] \odot [\gamma_2]^{-1} = [\text{Ad}(\text{Evol}([\gamma_2]))(\gamma_1 - \gamma_2)]$$

and

$$[\gamma]^{-1} = -[\text{Ad}(\text{Evol}([\gamma]))(\gamma)]$$

for all $[\gamma], [\gamma_1], [\gamma_2] \in \mathcal{E}([0, 1], \mathfrak{g})$. For fixed $[\gamma_2] \in \mathcal{E}([0, 1], \mathfrak{g})$, abbreviate $\eta := \text{Evol}([\gamma_2])^{-1}$. Then right translation with $[\gamma_2]$ in $(\mathcal{E}([0, 1], \mathfrak{g}), \odot)$ is the affine-linear map $\rho_{[\gamma_2]}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow \mathcal{E}([0, 1], \mathfrak{g})$,

$$[\gamma_1] \mapsto [\gamma_1] \odot [\gamma_2] = [\text{Ad}(\eta)\gamma_1] + [\gamma_2] = \tilde{f}(\eta, [\gamma_1]) + [\gamma_2]$$

with $f: G \times \mathfrak{g} \rightarrow \mathfrak{g}$, $f(x, y) := \text{Ad}_x(y)$ and

$$\tilde{f}: C([0, 1], G) \times \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow \mathcal{E}([0, 1], \mathfrak{g}), \quad (\sigma, [\tau]) \mapsto [f \circ (\sigma, \tau)].$$

The pushforward axioms and locality axiom entail that \tilde{f} is smooth (see Lemma 5.10) and hence continuous. As a consequence, the affine-linear map $\rho_{[\gamma_2]}$ is continuous and hence homeomorphism (as $\rho_{[\gamma_2]}^{-1}$ is continuous by the same argument). If G is a Lie group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then the affine \mathbb{K} -linear map $\rho_{[\gamma_2]}$ is a \mathbb{K} -analytic diffeomorphism $\mathcal{E}([0, 1], \mathfrak{g}) \rightarrow \mathcal{E}([0, 1], \mathfrak{g})$.

Definition 5.35 Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act smoothly on $AC_\mathcal{E}$. Let G be a local Lie group with global chart modelled on such a space, with Lie algebra \mathfrak{g} and neutral element e . Let D_G be the domain of the multiplication of G and $D_{\tilde{G}} = AC_\mathcal{E}([0, 1], D_G)$ be the domain of the multiplication in $AC_\mathcal{E}([0, 1], G)$. If G is locally \mathcal{E} -semiregular, then

$$\delta^\ell: AC_\mathcal{E}([0, 1], G)_* \rightarrow \mathcal{E}([0, 1], \mathfrak{g})$$

is an injective smooth map with $\delta^\ell(e) = 0$ whose image contains an open 0-neighbourhood $\Omega \subseteq \mathcal{E}([0, 1], \mathfrak{g})$. If G is locally \mathcal{E} -semiregular and

$$\text{Evol}: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], G)$$

is continuous, then $W := \{\eta \in AC_{\mathcal{E}}([0, 1], G)_* : \delta^\ell(\eta), \delta^\ell(\eta^{-1}) \in \Omega\}$ is an open identity neighbourhood in $AC_{\mathcal{E}}([0, 1], G)$. After replacing Ω with its open subset $\delta^\ell(W)$, we may assume that $\Omega = \delta^\ell(W)$ and thus

$$\delta^\ell|_W: W \rightarrow \Omega$$

is a homeomorphism. Consider W as a local Lie group with $D_W := \{(\eta_1, \eta_2) \in D_{\tilde{G}} \cap (W \times W) : \eta_1 \eta_2 \in W\}$. We give Ω the local topological group structure which makes $\delta^\ell|_W$ an isomorphism of local topological groups, and write \odot for the local multiplication on Ω . Then \odot is given by formulas as in the Lie group case. In particular, for each $[\gamma_2] \in \Omega$, we have

$$\rho_{[\gamma_2]}([\gamma_1]) = [\text{Ad}(\text{Evol}([\gamma_2]))^{-1}\gamma_1 + \gamma_2]$$

for $[\gamma_1]$ in some open 0-neighbourhood in Ω , and this is the restriction of an invertible affine-linear continuous map (with inverse of analogous form). If G is locally \mathcal{E} -regular, then we may assume that $\text{Evol}: \Omega \rightarrow W \subseteq AC_{\mathcal{E}}([0, 1], G)_*$ is smooth. Thus $\delta^\ell: W \rightarrow \Omega$ is a C^∞ -diffeomorphism and thus Ω (with \odot) is a smooth local Lie group.

Proposition 5.36 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let G be a \mathcal{E} -regular complex analytic Lie group modelled on a complex Fréchet space (resp., a sequentially complete complex (FEP)-space, resp., an integral complete complex locally convex space) E , with Lie algebra \mathfrak{g} . Then*

$$\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([0, 1], G)$$

is complex analytic. If G is a locally \mathcal{E} -regular complex analytic local Lie group modelled on E with a global chart and $\Omega \subseteq \mathcal{E}([0, 1], \mathfrak{g})$ an open 0-neighbourhood such that Evol is defined on Ω ,

$$\text{Evol}: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], G)$$

is smooth and $-[\text{Ad}(\text{Evol}([\gamma]))(\gamma)] \in \Omega$ for each $[\gamma] \in \Omega$, then Evol is complex analytic on Ω .

Proof. If G is a \mathcal{E} -regular complex analytic Lie group, set $\Omega := \mathcal{E}([0, 1], G)$; if G is a locally \mathcal{E} -regular complex analytic local Lie group, let $\Omega \subseteq \mathcal{E}([0, 1], \mathfrak{g})$ be an open 0-neighbourhood which is a smooth local Lie group with multiplication \odot as in Definition 5.35. Since Ω is open subset of the complex locally convex space $\mathcal{E}([0, 1], \mathfrak{g})$, we can consider it as an open complex analytic submanifold. The tangent map $T_0 \text{Evol}$ is complex linear, as it corresponds to the integration operator $\gamma \mapsto (t \mapsto \int_0^t \gamma(s) ds)$ (see Remark 5.30 (a)). Let $\rho_{[\gamma]^{-1}}$ be the right translation map $[\gamma_1] \mapsto [\gamma_1] \otimes [\gamma]^{-1}$ with $[\gamma] \in \Omega$ and r_η be the right translation map of $W \subseteq aC_{\mathcal{E}}([0, 1], G)_*$ with $\eta := \text{Evol}([\gamma])$. Since Evol is a smooth group homomorphism (resp., homomorphism of local Lie groups), we have

$$\text{Evol} = r_\eta \circ \text{Evol} \circ \rho_{[\gamma]}^{-1}$$

on some open neighbourhood of $[\gamma]$ and hence

$$T_{[\gamma]} \text{Evol} = T_e(r_\eta) \circ T_0(\text{Evol}) \circ T_{[\gamma]}(\rho_{[\gamma]}^{-1})$$

which is complex linear (recalling that $\rho_{[\gamma]^{-1}}$ is the restriction of a complex affine-linear continuous map). Thus Evol is a smooth map between complex analytic manifolds such that the tangent map $T_{[\gamma]} \text{Evol}$ is complex linear at each $[\gamma]$ in its domain. Therefore, Evol is complex analytic (cf. [19]). \square

Lemma 5.37 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let G be a real analytic local Lie group which is an open subset of a Fréchet space (resp., sequentially complete (FEP)-space, resp., integral complete locally convex space) E , with Lie algebra \mathfrak{g} . Let $\tilde{G} \subseteq E_{\mathbb{C}}$ be an open subset which is a complex analytic local Lie group with $G \subseteq \tilde{G}$, such that the inclusion map $G \rightarrow \tilde{G}$ is a homomorphism of real analytic local Lie groups. Then the following conditions are equivalent:*

- (a) *G is locally \mathcal{E} -regular and there is an open 0-neighbourhood Ω in $\mathcal{E}([0, 1], \mathfrak{g})$ such that each $[\gamma] \in \Omega$ has an evolution $\text{Evol}_G(\gamma)$ in G and*

$$\text{Evol}_G: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], G)$$

is real analytic.

(b) \tilde{G} is locally \mathcal{E} -regular.

Proof. After shrinking G and \tilde{G} , we may assume that complex conjugation $E_{\mathbb{C}} \rightarrow E_{\mathbb{C}}, x + iy \mapsto x - iy$ restricts to an antiholomorphic automorphism $\sigma: \tilde{G} \rightarrow \tilde{G}$ of the complex analytic local Lie group \tilde{G} and

$$G = \{z \in \tilde{G}: \sigma(z) = z\} = \tilde{G} \cap E.$$

Hence G is a real analytic submanifold of \tilde{G} and

$$AC_{\mathcal{E}}([0, 1], G) = AC_{\mathcal{E}}([0, 1], \tilde{G}) \cap AC_{\mathcal{E}}([0, 1], E)$$

is a real analytic submanifold of $AC_{\mathcal{E}}([0, 1], \tilde{G})$. If \tilde{G} is \mathcal{E} -regular, then we have a complex analytic evolution

$$\text{Evol}_{\tilde{G}}: \tilde{\Omega} \rightarrow AC_{\mathcal{E}}([0, 1], \tilde{G})$$

(see Proposition 5.36). Then $\Omega := \tilde{\Omega} \cap \mathcal{E}([0, 1], \mathfrak{g})$ is an open 0-neighbourhood in $\mathcal{E}([0, 1], \mathfrak{g})$. For each $[\gamma] \in \Omega$, we have

$$\text{Evol}_G([\gamma]) = \text{Evol}_{\tilde{G}}([\gamma]) \in AC_{\mathcal{E}}([0, 1], G)$$

(cf. Remark 5.28). hence Evol_G has the complex analytic extension $\text{Evol}_{\tilde{G}}$ and thus Evol_G is real analytic.

If, conversely, G is locally \mathcal{E} -regular with $\text{Evol}_G: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], G)$ real analytic, then Evol_G has a complex analytic extension

$$(\text{Evol}_G)^{\sim}: \tilde{\Omega} \rightarrow AC_{\mathcal{E}}([0, 1], \tilde{G}).$$

After shrinking Ω and $\tilde{\Omega}$ if necessary, we then have that $(\text{Evol}_G)^{\sim}([\gamma]) = \text{Evol}_{\tilde{G}}([\gamma])$ for all $[\gamma] \in \tilde{\Omega}$ (cf. [34, Proposition 9.9] for an analogous discussion of local C^k -regularity). Hence \tilde{G} is locally \mathcal{E} -regular. \square

Proposition 5.38 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let G be a Lie group modelled on such a space, with Lie algebra \mathfrak{g} . If G is \mathcal{E} -semiregular, then the following holds:*

- (a) *If $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([0, 1], G)$ is continuous at 0, then Evol is continuous.*
- (b) *If $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([0, 1], G)$ is C^1 on some open 0-neighbourhood, then Evol is smooth and thus G is \mathcal{E} -regular.*
- (c) *If $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([0, 1], G)$ is continuous and the smooth homomorphisms from G to \mathcal{E} -regular Lie groups separate points on G , then G is \mathcal{E} -regular.*
- (d) *If G is a real analytic Lie group and $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([0, 1], G)$ is real analytic on some open 0-neighbourhood, then Evol is real analytic.*
- (e) *If $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow C([0, 1], G)$ is continuous at 0, then the map $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow C([0, 1], G)$ is continuous and $(\mathcal{E}([0, 1], \mathfrak{g}), \odot)$ is a topological group.*

Proof. (a) For $[\gamma] \in \mathcal{E}([0, 1], \mathfrak{g})$, let $\rho_{[\gamma]^{-1}}$ be right translation with $[\gamma]^{-1}$ in the group $(\mathcal{E}([0, 1], \mathfrak{g}), \odot)$ and r_{η} be right translation with $\eta := \text{Evol}([\gamma])$ in the Lie group $AC_{\mathcal{E}}([0, 1], G)$. Since Evol is a group homomorphism from $(\mathcal{E}([0, 1], \mathfrak{g}), \odot)$ to $AC_{\mathcal{E}}([0, 1], G)$, we have

$$\text{Evol} = r_{\eta} \circ \text{Evol} \circ \rho_{[\gamma]^{-1}}.$$

Since r_{η} and $\rho_{[\gamma]^{-1}}$ are continuous, we see that Evol will be continuous at $[\gamma]$ if Evol is continuous at 0 (cf. [34, Theorem D] for an analogous result in the case of C^k -semiregularity).

(b) Step 1. If Evol is C^1 on an open 0-neighbourhood $W \subseteq \mathcal{E}([0, 1], \mathfrak{g})$ and $[\gamma] \in \mathcal{E}([0, 1], \mathfrak{g})$, then $W \odot [\gamma]$ is an open neighbourhood of $[\gamma]$ in $\mathcal{E}([0, 1], \mathfrak{g})$ (since $\rho_{[\gamma]}$ is a C^{∞} -diffeomorphism) and the formula

$$\text{Evol}|_{W \odot [\gamma]} = r_{\eta} \circ \text{Evol}|_W \circ \rho_{[\gamma]^{-1}}|_{W \odot [\gamma]}$$

shows that $\text{Evol}|_{W \odot [\gamma]}$ is C^1 . Hence Evol is C^1 .

Step 2. The map

$$f: G \times \mathfrak{g} \rightarrow \mathfrak{g}, \quad g(x, y) := \text{Ad}_x(y)$$

is linear in its second argument and smooth. Hence

$$\tilde{f}: C([0, 1], G) \times \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow \mathcal{E}([0, 1], \mathfrak{g}), \quad \tilde{f}(\eta, [\gamma]) := [f \circ (\eta, \gamma)]$$

is smooth by Lemma 5.10. As a consequence, the group multiplication

$$AC_{\mathcal{E}}([0, 1], \mathfrak{g})^2 \rightarrow AC_{\mathcal{E}}([0, 1], \mathfrak{g}), \quad ([\gamma_1], [\gamma_2]) \mapsto \tilde{f}(\text{Evol}([\gamma_2])^{-1}, [\gamma_1]) + [\gamma_2]$$

is C^1 and also the inversion map

$$AC_{\mathcal{E}}([0, 1], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([0, 1], \mathfrak{g}), \quad [\gamma] \mapsto -\tilde{f}(\text{Evol}([\gamma]), [\gamma]).$$

An inductive argument as in the case of C^k -regularity in [34, Theorem E] now show that Evol is C^k for each $k \in \mathbb{N}$ and hence smooth.

(c) We can repeat the proof of [34, Theorem F].

(d) Replace ‘smooth’ with ‘real analytic’ in Step 1 from the proof of (b).

(e) Since $C([0, 1], G)$ is a topological group when endowed with the compact-open topology, we can argue as in the proof of (a). The continuity of the group operations of $\mathcal{E}([0, 1], \mathfrak{g})$ follow with the Pushforward Axiom (P2). \square

Proposition 5.39 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, has the subdivision property, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let G be a real analytic Lie group modelled on such a space and \widetilde{W} be a complex analytic local Lie group with global chart, such that \widetilde{W} is a complexification of some open symmetric identity neighbourhood $W \subseteq G$ with global chart and the inclusion map $W \rightarrow \widetilde{W}$ is a homomorphism of real analytic local Lie groups. Then the following conditions are equivalent:*

- (a) G is \mathcal{E} -regular and $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([0, 1], G)$ is real analytic;
- (b) \widetilde{W} is locally \mathcal{E} -regular.

Proof. If G is \mathcal{E} -regular and $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow G$ is real analytic, then W is locally \mathcal{E} -regular and $\text{Evol}_W := \text{Evol}|_{\Omega}: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], W)$ is real analytic for some open 0-neighbourhood $\Omega \subseteq \mathcal{E}([0, 1], \mathfrak{g})$. Hence \widetilde{W} is locally \mathcal{E} -regular, by Lemma 5.37.

If, conversely, \widetilde{W} is locally \mathcal{E} -regular, then W is locally \mathcal{E} -regular with real analytic evolution $\text{Evol}_W: \Omega \rightarrow AC_{\mathcal{E}}([0, 1], W)$ on an open 0-neighbourhood $\Omega|_{\text{sub}\mathcal{E}}([0, 1], \mathfrak{g})$, by Lemma 5.37. Hence G is locally \mathcal{E} -regular. As \mathcal{E} has the subdivision property, we deduce with Proposition 5.25 that G is \mathcal{E} -regular. Let $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([0, 1], G)$ be the evolution map. Since $\text{Evol}|_{\Omega} = \text{Evol}_W$ is real analytic, Evol is real analytic by Proposition 5.38 (d). \square

6 Banach-Lie groups are L^1 -regular

In this section, we prove Theorem C and related results.

Definition 6.1 Let P be a set, $(F, \|\cdot\|)$ be a Banach space and $U \subseteq F$ be a subset. We say that a mapping

$$f: P \times U \rightarrow F$$

defines a uniform family of contractions if there exists $\theta \in [0, 1[$ such that

$$(\forall p \in P)(\forall y, z \in U) \quad \|f(p, z) - f(p, y)\| \leq \theta \|z - y\|.$$

We recall from [32] (cf. also [33]):

Lemma 6.2 *Let E be a locally convex space, $(F, \|\cdot\|)$ be a Banach space, $P \subseteq E$ and $U \subseteq F$ be open sets, $k \in \mathbb{N}_0 \cup \{\infty\}$ and*

$$f: P \times U \rightarrow F$$

be a C^k -map. Let $Q \subseteq P$ be the set of all $p \in P$ such that $f_p := f(p, \cdot): U \rightarrow F$, $y \mapsto f(p, y)$ has a fixed point x_p . Then x_p (if it exists) is unique, Q is open in P and the map

$$\psi: Q \rightarrow U, \quad p \mapsto x_p$$

taking a parameter $p \in Q$ to the fixed point of f_p is C^k . \square

Lemma 6.3 *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and G be a local \mathbb{K} -analytic Lie group modelled on a Banach space, which admits a global chart. Let $\mathfrak{g} = L(G)$. Then G is locally L^1 -regular and there is an open 0-neighbourhood $\Omega \subseteq L^1([0, 1], \mathfrak{g})$ on which Evol is defined and such that*

$$\text{Evol}: \Omega \rightarrow AC_{L^1}([0, 1], G)$$

is \mathbb{K} -analytic.

Proof. We may assume that G is an open subset of its modelling space E and $e = 0$. We identify $L(G) = T_0(G) = \{0\} \times E$ with E . Let $D_G \subseteq G \times G$ be the domain of the multiplication

$$\mu: D_G \rightarrow G.$$

We shall use

$$g: G \times E \rightarrow E, \quad g(x, y) := d\mu(x, 0; 0, y)$$

and the second differential

$$d^{(2)}\mu: D_G \times (E \times E) \times (E \times E) \rightarrow E.$$

Endow $E \times E$ with the maximum norm, $(x, y) \mapsto \max\{\|x\|, \|y\|\}$. Since $d^{(2)}\mu$ is continuous, there is an open convex 0-neighbourhood $U \subseteq G$ and $r > 0$ such that

$$d^{(2)}\mu(U \times (\overline{B}_r^{E \times E}(0))^2) \subseteq \overline{B}_1^E(0).$$

Therefore the continuous bilinear map $d^{(2)}\mu(x, 0, \cdot): (E \times E)^2 \rightarrow E$ has operator norm

$$\|d^{(2)}\mu(x, 0, \cdot)\|_{op} \leq \frac{1}{r^2},$$

for each $x \in U$. Given $\theta \in]0, 1[$, consider the open ball

$$P := \{[\gamma] \in L^1([0, 1], E): \int_0^1 \|\gamma(t)\| dt < r^2\theta\}$$

around 0 in $L^1([0, 1], E)$ and the map

$$f: L^1([0, 1], E) \times C([0, 1], U) \rightarrow C([0, 1], E), \quad f([\gamma], \eta)(t) := \int_0^t g(\eta(s), \gamma(s)) ds.$$

Note that

$$J: L^1([0, 1], E) \rightarrow C([0, 1], E), \quad J([\gamma])(t) := \int_0^t \gamma(s) ds$$

is a continuous linear map with operator norm $\|J\|_{op} \leq 1$. Now

$$\widetilde{g}: C([0, 1], U) \times L^1([0, 1], E) \rightarrow L^1([0, 1], E), \quad (\eta, [\gamma]) \mapsto [g \circ (\eta, \gamma)]$$

is a smooth map (as L^1 satisfies the pushforward axiom) and

$$f = J \circ \widetilde{g}.$$

Hence f is smooth. Moreover, $f|_{P \times C([0,1], U)}$ defines a uniform family of contractions. In fact, if $[\gamma] \in P$ and $\eta_1, \eta_2 \in C([0, 1], U)$, then

$$\begin{aligned}
& \|g(\eta_2(s), \gamma(s)) - g(\eta_1(s), \gamma(s))\| \\
&= \left\| \int_0^1 dg(\eta_1(s) + \tau(\eta_2(s) - \eta_1(s)), \gamma(s), \eta_2(s) - \eta_1(s), 0) d\tau \right\| \\
&\leq \int_0^1 \|dg(\eta_1(s) + \tau(\eta_2(s) - \eta_1(s)), \gamma(s), \eta_2(s) - \eta_1(s), 0)\| d\tau \\
&\leq \frac{1}{r^2} \|\gamma(s)\| \|\eta_2(s) - \eta_1(s)\| \leq \frac{1}{r^2} \|\gamma(s)\| \|\eta_2 - \eta_1\|_\infty
\end{aligned}$$

for all $s \in [0, 1]$, using that

$$\begin{aligned}
\|dg(x_\tau, \gamma(s), \eta_2(s) - \eta_1(s), 0)\| &= \|d^{(2)}\mu(x_\tau, 0, 0, \gamma(s), \eta_2(s) - \eta_1(s), 0)\| \\
&\leq \|d^{(2)}\mu(x_\tau, 0, \cdot)\|_{op} \|\gamma(s)\| \|\eta_2(s) - \eta_1(s)\|
\end{aligned}$$

for each $\tau \in [0, 1]$, with $x_\tau := \eta_1(s) + \tau(\eta_2(s) - \eta_1(s))$. Hence

$$\begin{aligned}
\|\tilde{g}(\eta_2, [\gamma]) - \tilde{g}(\eta_1, [\gamma])\|_{L^1} &= \int_0^1 \|g(\eta_2(s), \gamma(s)) - g(\eta_1(s), \gamma(s))\| ds \\
&\leq \int_0^1 \frac{1}{r^2} \|\gamma(s)\| \|\eta_2 - \eta_1\|_\infty ds \\
&\leq \frac{\|[\gamma]\|_{L^1}}{r^2} \|\eta_2 - \eta_1\|_\infty \leq \theta \|\eta_2 - \eta_1\|_\infty
\end{aligned}$$

and thus

$$\|f([\gamma], \eta_2) - f([\gamma], \eta_1)\| \leq \|J\|_{op} \|\tilde{g}(\eta_2, [\gamma]) - \tilde{g}(\eta_1, [\gamma])\| \leq \theta \|\eta_2 - \eta_1\|_\infty.$$

Thus f defines a uniform family of contractions. By Lemma 6.2, the set Ω of all $[\gamma] \in P$ for which $f([\gamma], \cdot): C([0, 1], U) \rightarrow C([0, 1], E)$ has a fixed point $\psi([\gamma]) := \eta$ is open in P , and the map

$$\psi: \Omega \rightarrow C([0, 1], U), \quad [\gamma] \mapsto \psi([\gamma])$$

is smooth. Note that $f(0, 0) = 0$, i.e., $0 \in C([0, 1], U)$ is a fixed point of $f(0, \cdot)$. Hence $0 \in \Omega$ and thus Ω is an open 0-neighbourhood in $L^1([0, 1], E)$. Note that, for each $[\gamma] \in \Omega$, $\eta := \psi([\gamma])$ satisfies

$$\eta = f([\gamma], \eta) = J(\tilde{g}(\eta, [\gamma])), \tag{45}$$

whence $\eta \in \text{im}(J) \subseteq AC_{L^1}([0, 1], E)$. By (45), we have

$$\eta(t) = \int_0^t g(\eta(s), \gamma(s)) ds$$

for all $t \in [0, 1]$. Hence η is a Caratheodory solution to

$$\eta'(t) = g(\eta(s), \gamma(s)), \quad \eta(0) = 0.$$

In other words, η is a Caratheodory solution to

$$\eta'(t) = d\mu(\eta(s), 0, 0, \gamma(s)), \quad \eta(0) = 0$$

and thus $\eta = \text{Evol}([\gamma])$. Thus G is locally L^1 -regular.

After shrinking G , we may assume that the inclusion map from G into \tilde{G} is a homomorphism of real analytic local Lie groups for a complex analytic local Banach-Lie group \tilde{G} which is an open subset of $E_{\mathbb{C}} = E \oplus iE$. By the preceding, \tilde{G} is locally L^1 -regular. Hence

$$\text{Evol} = \psi: \Omega \rightarrow G$$

is real analytic (possibly after shrinking the open 0-neighbourhood $\Omega \subseteq \mathcal{E}([0, 1], E)$), by Lemma 5.37. \square

We deduce the following result (which subsumes Theorem C):

Proposition 6.4 *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and G be a \mathbb{K} -analytic Banach-Lie group with Lie algebra $\mathfrak{g} = L(G)$. Then G is L^p -regular for each $p \in [1, \infty]$ and*

$$\text{Evol}: L^p([0, 1], \mathfrak{g}) \rightarrow AC_{L^p}([0, 1], G)$$

is \mathbb{K} -analytic.

Proof. By Lemma 6.3, G is locally L^1 -regular, whence G is L^1 -regular (by Proposition 5.25) and hence L^p -regular (see Theorem A). Let

$$\text{Evol}_G: L^p([0, 1], \mathfrak{g}) \rightarrow AC_{L^p}([0, 1], G)$$

be the evolution map. To see that Evol_G is not only smooth, but real analytic, let $U \subseteq G$ be an open symmetric identity neighbourhood on which a chart is defined and which injects into a complex analytic local Lie group \tilde{U}

with global chart, such that U is the fixed point set of an antiholomorphic involution and the inclusion map $U \rightarrow \tilde{U}$ an isomorphism of local groups onto the latter. Then \tilde{U} is locally L^1 -regular (by Lemma 6.3) and thus \tilde{U} is locally L^p -regular (see Corollary 5.21 and Remark 5.23). As a consequence, U (and hence also G) is locally L^p -regular and

$$\text{Evol}_G|_{\Omega}$$

is real analytic on some open 0-neighbourhood $\Omega \subseteq L^p([0, 1], \mathfrak{g})$ (Lemma 5.37). Hence Evol_G is real analytic, by Proposition 5.38 (d). \square

7 Measurable regularity for projective limits

We describe situations where measurable regularity properties pass from the steps G_n of a projective system

$$\cdots \rightarrow G_2 \rightarrow G_1$$

of Lie groups to the projective limit $G = \varprojlim G_n$. As a tool, we first discuss projective limits of Lebesgue spaces and spaces of absolutely continuous functions. The proofs of Lemmas 7.1–7.3 have been relegated to an appendix (Appendix B).

Lemma 7.1 *Let (X, Σ, μ) be a measure space, $((E_n)_{n \in \mathbb{N}}, (\phi_{n,m})_{n \leq m})$ be a projective system of Fréchet spaces E_n and continuous linear mappings $\phi_{n,m}: E_m \rightarrow E_n$ for $n \leq m$. Let E be a Fréchet space such that $E = \varprojlim E_n$, with the limit maps $\phi_n: E \rightarrow E_n$. Then*

$$L^p(X, \mu, E) = \varprojlim L^p(X, \mu, E_n)$$

for each $p \in [1, \infty]$, with bonding maps $L^p(X, \mu, \phi_{n,m})$ and the limit maps $L^p(X, \mu, \phi_n)$.

Lemma 7.2 *Let (X, Σ, μ) be a measure space, $((E_n)_{n \in \mathbb{N}}, (\phi_{n,m})_{n \leq m})$ be a projective system of locally convex spaces E_n and continuous linear maps $\phi_{n,m}: E_m \rightarrow E_n$ for $n \leq m$ in \mathbb{N} . Let E be a locally convex space such that $E = \varprojlim E_n$, with the limit maps $\phi_n: E \rightarrow E_n$. Then*

$$L_{rc}^\infty(X, \mu, E) = \varprojlim L_{rc}^\infty(X, \mu, E_n)$$

with bonding maps $L_{rc}^\infty(X, \mu, \phi_{n,m})$ and the limit maps $L_{rc}^\infty(X, \mu, \phi_n)$.

Lemma 7.3 *Let $a < b$ be real numbers, $((E_n)_{n \in \mathbb{N}}, (\phi_{n,m})_{n \leq m})$ be a projective system of Fréchet spaces E_n and continuous linear maps $\phi_{n,m}: E_n \rightarrow E_m$ for $n \leq m$ in \mathbb{N} . Let E be a Fréchet space such that $E = \varprojlim E_n$, with the limit maps $\phi_n: E \rightarrow E_n$. If $\phi_{n,m}(E_m)$ is dense in E_n for all $n, m \in \mathbb{N}$ such that $n \leq m$, then $R([a, b], E) = \varprojlim R([a, b], E_n)$ with bonding maps $R([a, b], \phi_{n,m})$ and the limit maps $R([a, b], \phi_n)$.*

Lemma 7.4 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., integral complete locally convex spaces) and $a < b$ be real numbers. Let $((E_n)_{n \in \mathbb{N}}, (\phi_{n,m})_{n \leq m})$ be a projective system of Fréchet spaces (resp., integral complete locally convex spaces) E_n and continuous linear maps $\phi_{n,m}: E_n \rightarrow E_m$ for $n \leq m$ in \mathbb{N} . Let E be a locally convex space such that $E = \varprojlim E_n$, with the limit maps $\phi_n: E \rightarrow E_n$. We assume that*

$$\mathcal{E}([a, b], E) = \varprojlim \mathcal{E}([a, b], E_n)$$

as a locally convex space, with bonding maps $\mathcal{E}([a, b], \phi_{n,m})$ and the limit maps $\mathcal{E}([a, b], \phi_n)$. Then

$$AC_{\mathcal{E}}([a, b], E) = \varprojlim AC_{\mathcal{E}}([a, b], E_n)$$

as a locally convex space, with bonding maps $AC_{\mathcal{E}}([a, b], \phi_{n,m})$ and the limit maps $AC_{\mathcal{E}}([a, b], \phi_n)$.

Proof. For each $n \in \mathbb{N}$, the map

$$\psi_n: AC_{\mathcal{E}}([a, b], E_n) \rightarrow E_n \times \mathcal{E}([a, b], E_n), \quad \eta \mapsto (\eta(a), \eta')$$

is an isomorphism of topological vector spaces and so is the corresponding map $\psi: AC_{\mathcal{E}}([a, b], E) \rightarrow E \times \mathcal{E}([a, b], E)$. Hence

$$AC_{\mathcal{E}}([a, b], E) \cong E \times \mathcal{E}([a, b], E) = \varprojlim (E_n \times \mathcal{E}([a, b], E_n)) \cong \varprojlim AC_{\mathcal{E}}([a, b], E_n).$$

In more detail, let us make the bonding maps and limit maps explicit which are involved, to ensure the final conclusion of the lemma. First, note that the spaces

$$E_n \times \mathcal{E}([a, b], E_n)$$

form a projective system together with the bonding maps $\phi_{n,m} \times \mathcal{E}([a, b], \phi_{n,m})$. By the compatibility of projective limits and direct products, we have that

$$E \times \mathcal{E}([a, b], E) = \varprojlim (E_n \times \mathcal{E}([a, b], E_n))$$

for this system, with the limit maps $\phi_n \times \mathcal{E}([a, b], \phi_n)$. Next, note that the locally convex spaces

$$AC_{\mathcal{E}}([a, b], E_n)$$

form a projective system with the bonding maps $AC_{\mathcal{E}}([a, b], \phi_{n,m})$. Since

$$(\phi_{n,m} \times \mathcal{E}([a, b], \phi_{n,m})) \circ \psi_m = \psi_n \circ AC_{\mathcal{E}}([a, b], \phi_{n,m})$$

and thus

$$\psi_n^{-1} \circ (\phi_{n,m} \times \mathcal{E}([a, b], \phi_{n,m})) = AC_{\mathcal{E}}([a, b], \phi_{n,m}) \circ \psi_m^{-1},$$

we have

$$E \times \mathcal{E}([a, b], E) = \varprojlim AC_{\mathcal{E}}([a, b], E_n)$$

for the preceding projective system, with the limit maps $\psi_n^{-1} \circ (\phi_n \times \mathcal{E}([a, b], \phi_n))$. Since $\psi: AC_{\mathcal{E}}([a, b], E) \rightarrow E \times \mathcal{E}([a, b], E)$ is an isomorphism of topological vector spaces, we conclude that

$$AC_{\mathcal{E}}([a, b], E) = \varprojlim AC_{\mathcal{E}}([a, b], E_n)$$

for the preceding projective system, with the desired limit maps

$$\psi_n^{-1} \circ (\phi_n \times \mathcal{E}([a, b], \phi_n)) \circ \psi = AC_{\mathcal{E}}([a, b], \phi_{n,m}).$$

We used that $(\phi_n \times \mathcal{E}([a, b], \phi_n)) \circ \psi = \psi_n \circ AC_{\mathcal{E}}([a, b], \phi_{n,m})$ as both maps take $\eta \in AC_{\mathcal{E}}([a, b], E)$ to $(\phi_n(\eta(a)), \phi_n \circ \eta') = (\phi_n(\eta(a)), (\phi_n \circ \eta)')$. \square

Definition 7.5 Let $((G_n)_{n \in \mathbb{N}}, (q_{n,m})_{n \leq m})$ be a projective system of Lie groups G_n modelled on locally convex spaces E_n and smooth group homomorphisms $q_{n,m}: G_m \rightarrow G_n$. Let G be a Lie group modelled on a locally convex space E such that

$$G = \varprojlim G_n$$

for the above projective system as a set, with limit maps $q_n: G \rightarrow G_n$ which are smooth group homomorphisms. We say that a chart $\phi: U \rightarrow E$ of G with $e \in U$ is a *projective limit chart* if

(a) There exist continuous linear maps

$$\alpha_n: E \rightarrow E_n \quad \text{and} \quad \alpha_{n,m}: E_m \rightarrow E_n$$

such that $((E_n)_{n \in \mathbb{N}}, (\alpha_{n,m})_{n \leq m})$ is a projective system of locally convex spaces and

$$E = \varprojlim E_n$$

for this system as a locally convex space, with the limit maps α_n ; and

(b) G_n is modelled on E_n and there exist charts $\phi_n: U_n \rightarrow V_n$ of G_n such that

$$\phi_{n,m}(U_m) \subseteq U_n \quad \text{and} \quad \alpha_{n,m}(V_m) \subseteq V_n$$

for all $n, m \in \mathbb{N}$ with $n \leq m$,

$$U = \bigcap_{n \in \mathbb{N}} q_n^{-1}(U_n), \quad V = \bigcap_{n \in \mathbb{N}} \alpha_n^{-1}(V_n),$$

$$\phi_n \circ q_{n,m}|_{U_m} = \alpha_{n,m} \circ \phi_m \quad \text{for all } n, m \in \mathbb{N} \text{ with } n \leq m,$$

$$q_n(U) \subseteq U_n \quad \text{and} \quad \alpha_n(V) \subseteq V_n \quad \text{for all } n \in \mathbb{N},$$

and

$$\alpha_n \circ \phi = \phi_n \circ q_n|_U \quad \text{for all } n \in \mathbb{N}.$$

We mention that existence of projective limit charts can also be characterized as follows (see Appendix B for the straightforward proof).

Lemma 7.6 *A Lie group $G = \varprojlim G_n$ as in Definition 7.5 admits a projective limit chart if and only if*

$$L(G) = \varprojlim L(G_n)$$

as a locally convex space with the bonding maps $L(q_{n,m})$ and limit maps $L(q_n)$, and there exist C^∞ -diffeomorphisms

$$\psi: U \rightarrow W \quad \text{and} \quad \psi_n: U_n \rightarrow W_n$$

from open identity neighbourhoods $U \subseteq G$ and $U_n \subseteq G_n$ onto open sets $W \subseteq L(G)$ and $W_n \subseteq L(G_n)$, respectively, such that

$$d\psi|_{L(G)} = \text{id}_{L(G)}, \quad d\psi_n|_{L(G_n)} = \text{id}_{L(G_n)} \quad \text{for all } n \in \mathbb{N},$$

$$q_{n,m}(U_m) \subseteq U_n \quad \text{and} \quad L(q_{n,m})(W_m) \subseteq W_n$$

for all $n, m \in \mathbb{N}$ with $n \leq m$,

$$U = \bigcap_{n \in \mathbb{N}} q_n^{-1}(U_n), \quad W = \bigcap_{n \in \mathbb{N}} L(q_n)^{-1}(W_n),$$

$$\psi_n \circ q_{n,m}|_{U_m} = L(q_{n,m}) \circ \psi_m \quad \text{for all } n, m \in \mathbb{N} \text{ with } n \leq m, \quad (46)$$

$$q_n(U) \subseteq U_n \quad \text{and} \quad L(q_n)(W) \subseteq W_n \quad \text{for all } n \in \mathbb{N} \quad (47)$$

and

$$L(q_n) \circ \psi = \psi_n \circ q_n|_U \quad \text{for all } n \in \mathbb{N}. \quad (48)$$

Remark 7.7 After shrinking U and the U_n , we can always achieve that $U = U^{-1}$ and $U_n = U_n^{-1}$ are symmetric identity neighbourhoods for all $n \in \mathbb{N}$ in Definition 7.5.

[In fact, $q_n(U \cap U^{-1}) = q_n(U) \cap q_n(U)^{-1} \subseteq U_n \cap U_n^{-1}$ and thus

$$q_n(U') \subseteq U'_n$$

for all $n \in \mathbb{N}$ if we define $U' := U \cap U^{-1}$ and $U'_n := U_n \cap U_n^{-1}$. We also set $V' := \phi(U')$ and $V'_n := \phi(U'_n)$. Then

$$\alpha_n(V') = \alpha_n(\phi(U')) = \phi_n(q_n(U')) \subseteq \phi_n(U'_n) = V'_n \quad (49)$$

for $n \in \mathbb{N}$. Likewise,

$$q_{n,m}(U'_m) \subseteq q_{n,m}(U_m) \cap q_{n,m}(U_m)^{-1} \subseteq U_n \cap U_n^{-1} = U'_n$$

for $n \leq m$ in \mathbb{N} and thus

$$\alpha_{n,m}(V'_m) = \alpha_{n,m}(\phi_m(U'_m)) = \phi_n(q_{n,m}(U'_m)) \subseteq \phi_n(U'_n) = V'_n.$$

Realizing the projective limit as a subgroup of $\prod_{n \in \mathbb{N}} G_n$ as usual, we see that $U = \bigcap_{n \in \mathbb{N}} q_n^{-1}(U_n)$ entails

$$U' = \bigcap_{n \in \mathbb{N}} q_n^{-1}(U'_n).$$

By (49), we have

$$V' \subseteq \bigcap_{n \in \mathbb{N}} \alpha_n^{-1}(V'_n).$$

To see that equality holds, let $y \in \bigcap_{n \in \mathbb{N}} \alpha_n^{-1}(V'_n)$. Then $y \in V$. Abbreviate $x := \phi^{-1}(y)$. For each $n \in \mathbb{N}$, we have that

$$\phi_n(q_n(x)) = \alpha_n(\phi(x)) = \alpha_n(y) \in V'_n$$

and hence $q_n(x) \in U'_n$. Thus $x \in \bigcap_{n \in \mathbb{N}} q_n^{-1}(U'_n) = U'$ and $y = \phi(x) \in V'$.

Proposition 7.8 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, has the subdivision property, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let $((G_n)_{n \in \mathbb{N}}, (q_{n,m})_{n \leq m})$ be a projective system of Lie groups G_n modelled on Fréchet spaces (resp., integral complete locally convex spaces) E_n and smooth group homomorphisms $q_{n,m}: G_m \rightarrow G_n$. Let G be a Lie group modelled on a locally convex space E such that*

$$G = \varprojlim G_n$$

for the above projective system as a set, with limit maps $q_n: G \rightarrow G_n$ which are smooth group homomorphisms. Assume that

- (a) *G admits a projective limit chart in the sense of Definition 7.5;*
- (b) *G_n is \mathcal{E} -regular for each $n \in \mathbb{N}$; and*
- (c) *$\mathcal{E}([0, 1], L(G)) = \varprojlim \mathcal{E}([0, 1], L(G_n))$ with respect to the bonding maps $\mathcal{E}([0, 1], L(q_{n,m}))$ and limit maps $\mathcal{E}([0, 1], L(q_n))$.*
- (d) *U_n in Lemma 7.6 can be chosen such that $U_n = U_n^{-1}$ and*

$$\Omega := \bigcap_{n \in \mathbb{N}} \mathcal{E}([0, 1], L(q_n))^{-1}(\text{Evol}_{G_n}^{-1}(AC_{\mathcal{E}}([0, 1], U_n)))$$

is a 0-neighbourhood in $\mathcal{E}([0, 1], L(G))$.

Then also G is \mathcal{E} -regular.

Remark 7.9 By Lemma 7.1 and Lemma 7.2, condition (c) of Proposition 7.8 is automatically satisfied if $\mathcal{E} = L^p$ as a bifunctor on Fréchet spaces or $\mathcal{E} = L_{rc}^{\infty}$ as a bifunctor to integral complete locally convex spaces. If $\mathcal{E} = R$ as a bifunctor to Fréchet spaces and $L(q_{n,m})$ has dense image for all positive integers $n \leq m$, then condition (c) of Proposition 7.8 is satisfied by Lemma 7.3.

Before proving Proposition 7.8, let us spell out simple situations in which condition (d) of the proposition is satisfied:

Lemma 7.10 *Condition (d) from Proposition 7.8 is automatically satisfied in the following situations:*

- (i) $L(q_{n,m})$ and $q_{n,m}$ are injective for all positive integers $n \leq m$ (whence also $L(q_n)$ is injective on $L(G) = \varprojlim L(G_n)$ and q_n is injective on $G = \varprojlim G_n$). Or:
- (ii) $U_n = G_n$ for each $n \in \mathbb{N}$.

Proof. In the situation of (ii), we simply have $\Omega = \mathcal{E}([0, 1], L(G))$. To prove (i), after identifying $L(G_n)$ with a vector subspace of $L(G_1)$ by means of the injective linear map $L(q_{1,n})$, we may assume that

$$L(G_1) \supseteq L(G_2) \supseteq \cdots$$

and $L(q_{n,m})$ is the inclusion map for all integers $n \leq m$. We identify $L(G)$ with the vector subspace $\bigcap_{n \in \mathbb{N}} L(G_n)$ of $L(G_1)$. Then also $L(q_n)$ becomes the inclusion map for each $n \in \mathbb{N}$. In the situation of Lemma 7.6, we then have $U_n \supseteq U_m$ for all positive integers $n \leq m$ and $U = \bigcap_{n \in \mathbb{N}} U_n$.

By the definition of the topology on the projective limit $L(G)$, there is $k \in \mathbb{N}$ and a 0-neighbourhood $P \subseteq L(G_k)$ such that $L(G) \cap P = L(q_k)^{-1}(P) \subseteq U$. After passing to a cofinal subsequence, we may assume that $k = 1$. Thus

$$P \subseteq U \subseteq U_n \quad \text{for all } n \in \mathbb{N}$$

and $P = P \cap L(G_n) = L(q_{1,n})^{-1}(P)$ is an open 0-neighbourhood in $L(G_n)$ for each $n \in \mathbb{N}$. As a consequence, $Q_n := \psi_n^{-1}(P) \subseteq U_n$ is an open identity neighbourhood in G_n and $Q := \psi^{-1}(P)$ is an open identity neighbourhood in G . We have

$$\psi_n(q_n(Q)) = L(q_n)(\psi(Q)) = L(q_n)(P) = P$$

and thus

$$q_n(Q) = \psi_n^{-1}(P) = Q_n$$

for each $n \in \mathbb{N}$. Hence

$$Q \subseteq \bigcap_{n \in \mathbb{N}} q_n^{-1}(Q_n).$$

To see that

$$Q = \bigcap_{n \in \mathbb{N}} q_n^{-1}(Q_n), \quad (50)$$

let $x \in \bigcap_{n \in \mathbb{N}} q_n^{-1}(Q_n)$. Then $x \in \bigcap_{n \in \mathbb{N}} q_n^{-1}(U_n) = U$. We have

$$\psi(x) = L(q_n)(\psi(x)) = \psi_n(q_n(x)) \in \psi_n(Q_n) = P$$

for each $n \in \mathbb{N}$ and thus $x \in \psi^{-1}(P) = Q$. Likewise,

$$\psi_n(q_{n,m}(Q_m)) = L(q_{n,m})(\psi_m(Q_m)) = L(q_{n,m})(P) = P$$

and thus

$$q_{n,m}(Q_m) = Q_n \quad (51)$$

for all positive integers $n \leq m$. We claim that the open 0-neighbourhood

$$\begin{aligned} \Omega' &:= \{[\gamma] \in \mathcal{E}([0, 1], L(G)) : \text{Evol}_{G_1}([L(q_1) \circ \gamma]) \in AC_{\mathcal{E}}([0, 1], Q_1)\} \\ &= \mathcal{E}([0, 1], L(q_1))^{-1}(\text{Evol}_{G_1}^{-1}(AC_{\mathcal{E}}([0, 1], Q_1))) \end{aligned} \quad (52)$$

in $\mathcal{E}([0, 1], L(G))$ coincides with the subset

$$\Omega' := \bigcap_{n \in \mathbb{N}} \mathcal{E}([0, 1], L(q_n))^{-1}(\text{Evol}_{G_n}^{-1}(AC_{\mathcal{E}}([0, 1], Q_n)))$$

of the intersection Ω from Proposition 7.8 (d). If this is true, then indeed Ω is a 0-neighbourhood. To prove the claim, note first that Ω' is a subset of (52) by definition. To prove the converse inclusion, let $[\gamma] \in \mathcal{E}([0, 1], L(G))$ such that

$$\text{Evol}_{G_1}([L(q_1) \circ \gamma]) \in AC_{\mathcal{E}}([0, 1], Q_1).$$

Then

$$\begin{aligned} q_{1,n} \circ \text{Evol}_{G_n}([L(q_n) \circ \gamma]) &= \text{Evol}_{G_1}([L(q_{1,n}) \circ L(q_n) \circ \gamma]) \\ &= \text{Evol}_{G_1}([L(q_1) \circ \gamma]) \in AC_{\mathcal{E}}([0, 1], Q_1), \end{aligned}$$

showing that $\text{Evol}_{G_n}([L(q_n) \circ \gamma])$ takes its values in

$$q_{1,n}^{-1}(Q_1) = Q_n$$

(using (51) and the injectivity of $q_{1,n}$). Thus

$$\text{Evol}_{G_n}([L(q_n) \circ \gamma]) \in AC_{\mathcal{E}}([0, 1], Q_n) \subseteq AC_{\mathcal{E}}([0, 1], U_n)$$

for each $n \in \mathbb{N}$ and thus $[\gamma] \in \Omega'$. \square

Proof of Proposition 7.8. Let $\psi_n: U_n \rightarrow W_n$ and $\psi: U \rightarrow W$ be as in Lemma 7.6. We may assume that U and each U_n is a symmetric identity neighbourhood. Hence $AC_{\mathcal{E}}([0, 1], \psi)$ and $AC_{\mathcal{E}}([0, 1], \psi_n)$ are charts for $AC_{\mathcal{E}}([0, 1], G)$ and $AC_{\mathcal{E}}([0, 1], G_n)$, respectively. Since G_n is \mathcal{E} -regular, we have a smooth evolution map

$$\text{Evol}_{G_n}: \mathcal{E}([0, 1], L(G_n)) \rightarrow AC_{\mathcal{E}}([0, 1], G_n).$$

Since $AC_{\mathcal{E}}([0, 1], U_n)$ is an open identity neighbourhood in $AC_{\mathcal{E}}([0, 1], G_n)$, we deduce that

$$\Omega_n := (\text{Evol}_{G_n})^{-1}(AC_{\mathcal{E}}([0, 1], U_n))$$

is an open 0-neighbourhood in $\mathcal{E}([0, 1], L(G_n))$. Since

$$q_{n,m} \circ \text{Evol}_{G_m} = \text{Evol}_{G_m} \circ L(q_{n,m})$$

and $q_{n,m}(U_m) \subseteq U_n$, we deduce that

$$AC_{\mathcal{E}}([0, 1], L(q_{n,m}))(\Omega_m) \subseteq \Omega_n \quad \text{for all } n \leq m \text{ in } \mathbb{N}.$$

Hypothesis (c) implies that

$$AC_{\mathcal{E}}([0, 1], L(G)) = \varprojlim AC_{\mathcal{E}}([0, 1], L(G_n)) \quad (53)$$

using the bonding maps $AC_{\mathcal{E}}([0, 1], L(q_{n,m}))$ and limit maps $AC_{\mathcal{E}}([0, 1], L(q_n))$ (see Lemma 7.4). Define

$$\Omega := \{[\gamma] \in \mathcal{E}([0, 1], L(G)) : (\forall n \in \mathbb{N}) \mathcal{E}([0, 1], L(q_n))([\gamma]) \in \Omega_n\}.$$

If $[\gamma] \in \Omega$, then

$$\eta_n := \text{Evol}_{G_n}(\mathcal{E}([0, 1], L(q_n))([\gamma])) = \text{Evol}_{G_n}([L(q_n) \circ \gamma]) \in AC_{\mathcal{E}}([0, 1], G_n)$$

for each $n \in \mathbb{N}$. Then

$$q_{n,m} \circ \eta_m = \text{Evol}_{G_n}([L(q_{n,m}) \circ L(q_m) \circ \gamma]) = \text{Evol}_{G_n}([L(q_n) \circ \gamma]) = \eta_n$$

for all $n \leq m$ in \mathbb{N} , whence there is a unique map

$$\eta: [0, 1] \rightarrow U \subseteq G$$

such that $q_n \circ \eta = \eta_n$ for all $n \in \mathbb{N}$. Define

$$\zeta := \psi \circ \eta \quad \text{and} \quad \zeta_n := \psi_n \circ \eta_n \quad \text{for } n \in \mathbb{N}.$$

Then

$$L(q_{n,m}) \circ \zeta_m = L(q_{n,m}) \circ \psi_m \circ \eta_m = \psi_n \circ q_{n,m} \circ \eta_m = \psi_n \circ \eta_n = \zeta_n$$

for all $n \leq m$ in \mathbb{N} . By (53), there is $\theta \in AC_{\mathcal{E}}([0, 1], L(G))$ such that

$$L(q_n) \circ \theta = \zeta_n \quad \text{for all } n \in \mathbb{N}, \quad (54)$$

Now

$$L(q_n) \circ \zeta = L(q_n) \circ \psi \circ \eta = \psi_n \circ q_n \circ \eta = \psi_n \circ \eta_n = \zeta_n \quad (55)$$

for all $n \in \mathbb{N}$. As the maps $L(q_n)$ separate points on $L(G) = \varprojlim L(G_n)$ for $n \in \mathbb{N}$, we deduce from (54) and (55) that

$$\zeta = \theta \in AC_{\mathcal{E}}([0, 1], L(G)). \quad (56)$$

Hence $\eta = \phi^{-1} \circ \zeta \in AC_{\mathcal{E}}([0, 1], G)$. Since

$$\mathcal{E}([0, 1], L(q_n))(\delta^\ell(\eta)) = \delta^\ell(q_n \circ \eta) = \delta^\ell(\eta_n) = \mathcal{E}([0, 1], L(q_n))([\gamma])$$

for each $n \in \mathbb{N}$ and the maps $\mathcal{E}([0, 1], L(q_n))$ separate points on $\mathcal{E}([0, 1], L(G)) = \varprojlim \mathcal{E}([0, 1], L(G_n))$, we deduce that

$$\delta^\ell(\eta) = [\gamma]$$

and thus $\eta = \text{Evol}([\gamma])$. If Ω is a 0-neighbourhood in $\mathcal{E}([0, 1], L(G))$, then G is locally \mathcal{E} -regular by the preceding. Moreover,

$$\begin{aligned} AC_{\mathcal{E}}([0, 1], L(q_n)) \circ AC_{\mathcal{E}}([0, 1], \psi) \circ \text{Evol}|_{\Omega^0} \\ = AC_{\mathcal{E}}([0, 1], \psi_n) \circ \text{Evol}_{G_n} \circ \mathcal{E}([0, 1], L(q_n))|_{\Omega^0} \end{aligned}$$

by (56) for each $n \in \mathbb{N}$, which is a smooth map (where Ω^0 denotes the interior of Ω). As a consequence, the map $AC_{\mathcal{E}}([0, 1], \psi) \circ \text{Evol}|_{\Omega^0}$ to

$$AC_{\mathcal{E}}([0, 1], L(G)) = \varprojlim AC_{\mathcal{E}}([0, 1], L(G_n))$$

is smooth and hence $\text{Evol}|_{\Omega^0}$ is smooth. Thus G is locally \mathcal{E} -regular and hence \mathcal{E} -regular, by Proposition 5.25. \square

Proposition 7.11 *Let M be a finite-dimensional smooth manifold, $K \subseteq M$ be a compact set and H be a Banach-Lie group. Then $C_K^\infty(M, H)$ is L^1 -regular.*

Proof. Let E be the modelling space of G and $\tau: U \rightarrow V$ be a chart for G with $\tau(e) = 0$, defined on a symmetric open identity neighbourhood $U \subseteq G$. Then

$$\phi := C_K^\infty(M, \tau): C_K^\infty(M, U) \rightarrow C_K^\infty(M, V) \subseteq C_K^\infty(M, E)$$

is a chart for $C_K^\infty(M, H)$ and

$$\phi_n := C_K^n(M, \tau): C_K^n(M, U) \rightarrow C_K^n(M, V) \subseteq C_K^n(M, E)$$

is a chart for $C_K^n(M, H)$, for each $n \in \mathbb{N}$ (cf. [20]). We have that

$$C_K^\infty(M, H) = \bigcap_{n \in \mathbb{N}} C_K^n(M, H) = \varprojlim C_K^n(M, H)$$

as a set (with the respective inclusion maps as the limit maps and bonding maps). Using the inclusion maps $\alpha_n: C_K^\infty(M, E) \rightarrow C_K^n(M, E)$ for $n \in \mathbb{N}$ and $\alpha_{n,m}: C_K^m(M, E) \rightarrow C_K^n(M, E)$ for positive integers $n \leq m$, we see that all conditions from Definition 7.5 are satisfied and thus ϕ is a projective limit chart. Hence condition (a) from Proposition 7.8 is satisfied and by Remark 7.9 and Lemma 7.10 (i), also conditions (c) and (d) are satisfied. Since every Banach-Lie group is L^1 -regular by Theorem C, also condition (b) is satisfied and hence $C_K^\infty(M, H)$ is L^1 -regular by Proposition 7.8. \square

Remark 7.12 The same argument shows that $C^\infty(M, H)$ is L^1 -regular if H is a Banach-Lie group and M a compact smooth manifold (possibly with boundary or corners).

Proposition 7.13 *Let $\pi: P \rightarrow M$ be a smooth principal bundle over a compact smooth manifold M , whose structure group is a Banach-Lie group H . Then the gauge group $\text{Gau}(P)$ is L^1 -regular.*

Proof. For some $m \in \mathbb{N}$, we can cover M by the interiors M_j^0 of compact submanifolds M_j of M with boundary for $j \in \{1, \dots, m\}$ (of full dimension), such that there is a smooth section $\sigma_j: M_j \rightarrow P$ for π . For all $i, j \in \{1, \dots, m\}$, there is a unique map $k_{i,j}: M_i \cap M_j \rightarrow H$ such that

$$\sigma_i(x)k_{i,j}(x) = \sigma_j(x) \quad \text{for all } x \in M_i \cap M_j.$$

Then $\text{Gau}(P)$ can be identified with the Lie subgroup S of

$$G := \prod_{j=1}^n C^\infty(M_j, H)$$

consisting of all $\gamma = (\gamma_j)_{j=1, \dots, m} \in G$ such that

$$(\forall i, j)(\forall x \in M_i \cap M_j) \gamma_i(x) = k_{i,j}(x) \gamma_j(x) k_{j,i}(x)$$

(see [73] or [65]). Each of the Lie groups $C^\infty(M_j, H)$ is L^1 -regular (see Remark 7.12), whence also the finite direct product $G = \prod_{j=1}^n C^\infty(M_j, H)$ is L^1 -regular (cf. Theorem G). For $i, j \in \{1, \dots, m\}$ and $x \in M_i \cap M_j$, the mappings

$$\alpha_{i,j,x}: G \rightarrow H, \quad \gamma \mapsto \gamma_i(x)$$

and

$$\beta_{i,j,k}: G \rightarrow H, \quad \gamma \mapsto k_{i,j}(x) \gamma_j(x) k_{j,i}(x)$$

are smooth group homomorphisms. Since

$$S = \{\gamma \in G: (\forall i, j \in \{1, \dots, m\})(\forall x \in M_i \cap M_j) \alpha_{i,j,x}(\gamma) = \beta_{i,j,x}(\gamma)\},$$

we deduce with Proposition 5.27 that S (and hence also $\text{Gau}(P)$) is L^1 -regular. \square

Also the following variant of Proposition 7.8 is useful.

Proposition 7.14 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., integral complete locally convex spaces) which satisfies the locality axiom, the push-forward axioms, has the subdivision property, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let $((G_n)_{n \in \mathbb{N}}, (q_{n,m})_{n \leq m})$ be a projective system of Lie groups G_n modelled on Fréchet spaces (resp., integral complete locally convex spaces) E_n and smooth group homomorphisms $q_{n,m}: G_m \rightarrow G_n$. Let G be a Lie group modelled on a locally convex space E such that*

$$G = \varprojlim G_n$$

for the above projective system as a set, with limit maps $q_n: G \rightarrow G_n$ which are smooth group homomorphisms. Assume that

- (a) *G admits a projective limit chart $\phi: U \rightarrow V$ determined by charts $\phi_n: U_n \rightarrow V_n$ as in Definition 7.5, such that $U = G$ and $U_n = G_n$;*

(b) G_n is \mathcal{E} -regular for each $n \in \mathbb{N}$; and

(c) $\mathcal{E}([0, 1], L(G)) = \varprojlim \mathcal{E}([0, 1], L(G_n))$ with respect to the bonding maps $\mathcal{E}([0, 1], L(q_{n,m}))$ and limit maps $\mathcal{E}([0, 1], L(q_n))$.

Then G is \mathcal{E} -regular.

Proof. After replacing G with V and G_n with V_n , we may assume that G is an open subset of E and G_n is an open subset of E_n for each $n \in \mathbb{N}$. Moreover, $q_n = \alpha_n|_G$ for each $n \in \mathbb{N}$ and $q_{n,m} = \alpha_{n,m}|_{G_m}$ for all positive integers $n \leq m$. We identify $L(G)$ with E and $L(G_n)$ with E_n . Then $L(q_n) = \alpha_n$ and $L(q_{n,m}) = \alpha_{n,m}$. If $[\gamma] \in \mathcal{E}([0, 1], E)$, we can form

$$\eta_n := \text{Evol}_{G_n}([\alpha_n \circ \gamma]) \in AC([0, 1], G_n) \subseteq AC([0, 1], E_n).$$

Then $\alpha_{n,m} \circ \eta_m = q_{n,m} \circ \eta_m = \text{Evol}_{G_n}([L(q_{n,m}) \circ L(q_m) \circ \gamma]) = \eta_m$ for all positive integers $n \leq m$. Since

$$AC([0, 1], E) = \varprojlim AC_{\mathcal{E}}([0, 1], E_n)$$

with the limit maps $AC_{\mathcal{E}}([0, 1], \alpha_n)$ and bonding maps $AC_{\mathcal{E}}([0, 1], \alpha_{n,m})$, we see that there is a unique $\Psi([\gamma]) := \eta$ in $AC_{\mathcal{E}}([0, 1], E)$ such that

$$\alpha_n \circ \eta = \eta_n$$

for all $n \in \mathbb{N}$. Since

$$AC_{\mathcal{E}}([0, 1], \alpha_n) \circ \Psi = \text{Evol}_{G_n} \circ \mathcal{E}([0, 1], \alpha_n)$$

is smooth for all $n \in \mathbb{N}$, we deduce that Ψ is smooth. In particular, Ψ is continuous and since $\Psi(0) = e \in AC_{\mathcal{E}}([0, 1], G)$ and $AC_{\mathcal{E}}([0, 1], G)$ is open in $AC_{\mathcal{E}}([0, 1], E)$, we deduce that

$$\Omega := \Psi^{-1}(AC_{\mathcal{E}}([0, 1], G))$$

is an open 0-neighbourhood in $\mathcal{E}([0, 1], E)$. As in the proof of Proposition 7.8, we see that $\Psi([\gamma]) = \text{Evol}([\gamma])$ for each $[\gamma] \in \Omega$. Since $\text{Evol}|_{\Omega} = \Psi|_{\Omega}$ is smooth, G is locally \mathcal{E} -regular and hence \mathcal{E} -regular, as we assume that \mathcal{E} has the subdivision property. \square

See, e.g., [21] for the notion of a continuous inverse algebra A and the fact that its group A^\times of invertible elements is an analytic Lie group. For the concept of locally m -convex topological algebra, see [50]. If a locally m -convex continuous inverse algebra is integral complete, then A^\times is C^0 -regular [37].

Proposition 7.15 *Let A be a continuous inverse algebra. If A is locally m -convex and a Fréchet space, then its unit group A^\times is L^1 -regular.*

Proof. Like every locally m -convex Fréchet algebra, A is a countable projective limit $A = \varprojlim A_n$ of Banach algebras A_n . Since A_n^\times is L^1 -regular by Theorem C and the identity maps on A^\times and A_n^\times are global charts, we deduce with Proposition 7.14 that A^\times is L^1 -regular. \square

Neeb and Wagemann [56] constructed a regular Lie group structure on the mapping group $C^\infty([0, 1], H)$, for each regular Lie group H .

Proposition 7.16 *$C^\infty(\mathbb{R}, H)$ is L^1 -regular for each Banach-Lie group H .*

Proof. Since $C^\infty(\mathbb{R}, H) = C^\infty(\mathbb{R}, H)_* \rtimes H$ where H is L^1 -regular by Theorem C, we deduce with Theorem G that $C^\infty(\mathbb{R}, H)$ will be L^1 -regular if we can show that

$$C^\infty(\mathbb{R}, H)_* := \{\eta \in C^\infty(\mathbb{R}, H) : \eta(0) = e\}$$

is L^1 -regular. Abbreviate $\mathfrak{h} := L(H)$. As shown in [56], the map

$$\phi : C^\infty(\mathbb{R}, H)_* \rightarrow C^\infty(\mathbb{R}, \mathfrak{h}), \quad \eta \mapsto \delta^\ell(\eta)$$

is a global chart for $C^\infty(\mathbb{R}, H)$. Likewise,

$$\phi_n : C^\infty([-n, n], H)_* \rightarrow C^\infty([-n, n], \mathfrak{h}), \quad \eta \mapsto \delta^\ell(\eta)$$

is a global chart for the Lie subgroup

$$C^\infty([-n, n], H)_* := \{\eta \in C^\infty([-n, n], H) : \eta(0) = e\}$$

of $C^\infty([-n, n], H)$. Now

$$C^\infty([-n, n], H)_* := \{\eta \in C^\infty([-n, n], H) : \alpha(\eta) = \beta(\eta)\}$$

with the smooth homomorphisms $\alpha, \beta : C^\infty([-n, n], H) \rightarrow H$, $\alpha(\eta) := \eta(0)$, $\beta(\eta) := e$. Since $C^\infty([-n, n], H)$ is L^1 -regular (see Remark 7.12), we deduce with Proposition 5.27 that $C^\infty([-n, n], H)$ is L^1 -regular. Note that

$$C^\infty(\mathbb{R}, \mathfrak{h}) = \varprojlim C^\infty([-n, n], \mathfrak{h})$$

as a locally convex space. Using Proposition 7.14, we find that $C^\infty([0, 1], H)_* = \varprojlim C^\infty([-n, n], H)$ is L^1 -regular. \square

8 Measurable regularity for weak direct products and some direct limits

8.1 Let $(H_j)_{j \in J}$ be a family of Lie groups H_j , with modelling space E_j . Let

$$E := \bigoplus_{j \in J} E_j := \left\{ (x_j)_{j \in J} \in \prod_{j \in J} E_j : x_j = 0 \text{ for all but finitely many } j \right\}$$

be the direct sum of the given locally convex spaces, endowed with the locally convex direct sum topology. Then

$$G := \bigoplus_{j \in J} H_j := \left\{ (x_j)_{j \in J} \in \prod_{j \in J} H_j : x_j = e \text{ for all but finitely many } j \right\}$$

is a group under pointwise multiplication. If $\phi_j : U_j \rightarrow V_j$ is a chart for H_j defined on an open identity neighbourhood $U_j = U_j^{-1}$ in H_j with $\phi(e) = 0$, then G can be given a Lie group structure modelled on E such that

$$\phi := \bigoplus_{j \in J} \phi_j : \bigoplus_{j \in J} U_j \rightarrow \bigoplus_{j \in J} V_j, \quad (x_j)_{j \in J} \mapsto (\phi_j(x_j))_{j \in J}$$

is a chart around the identity element (cf. [24]). Here

$$\bigoplus_{j \in J} U_j := \bigoplus_{j \in J} H_j \cap \prod_{j \in J} U_j \quad \text{and} \quad \bigoplus_{j \in J} V_j := \bigoplus_{j \in J} E_j \cap \prod_{j \in J} V_j.$$

The Lie group $\bigoplus_{j \in J} H_j$ is called the *weak direct product* of the family $(H_j)_{j \in J}$ of Lie groups.

Proposition 8.2 *If $(H_j)_{j \in J}$ is a family of L^1 -regular Lie groups H_j modelled on sequentially complete (FEP)-spaces, then also the weak direct product $G := \bigoplus_{j \in J} H_j$ is modelled on a sequentially complete (FEP)-space and L^1 -regular.*

Proof. Let E_j be the modelling space of G_j and $E := \bigoplus_{j \in J} E_j$, which is a sequentially complete (FEP)-space by Lemma 1.41 (a). Pick a chart $\phi_j : U_j \rightarrow V_j \subseteq E_j$ for H_j around e with $U_j = U_j^{-1}$ and $\phi_j(e) = 0$. Let $\phi := \bigoplus_{j \in J} \phi_j : U \rightarrow V$ be the corresponding chart of G , with $U := \bigoplus_{j \in J} U_j$ and $V := \bigoplus_{j \in J} V_j \subseteq E$. We identify $L(G)$ with E using the isomorphism

$d\phi|_{L(G)}$, and $L(H_j)$ with E_j using $d\phi_j|_{L(H_j)}$. If $F \subseteq J$ is a finite set, we consider

$$G_F := \prod_{j \in F} H_j$$

as a Lie subgroup of G and identify $L(G_F)$ with $\prod_{j \in F} E_j$. Let $\iota_F: G_F \rightarrow G$ be the inclusion map. With identifications as before, $L(\iota_F)$ is the inclusion map

$$\prod_{j \in F} E_j \rightarrow E.$$

If $[\gamma] \in L^1([0, 1], E)$, after changing the representative if necessary we may assume that $\gamma \in \mathcal{L}^1([0, 1], \prod_{j \in F} E_j)$ for some finite subset $F \subseteq J$ (see Lemma 1.41 (a)). Hence $\eta := \text{Evol}_{G_F}([\gamma])$ is defined and

$$\delta^\ell(\iota_F \circ \eta) = L(\iota_F) \circ \delta^\ell(\eta) = [\gamma]$$

entails that

$$\iota_F \circ \eta = \text{Evol}_G([\gamma]). \quad (57)$$

Let γ_j be the j -th component of γ . Then

$$\eta = (\text{Evol}_{G_j}([\gamma_j]))_{j \in F}. \quad (58)$$

Recall from Lemma 1.41 (a) that the summation map

$$\Sigma: \bigoplus_{j \in J} L^1([0, 1], E_j) \rightarrow L^1([0, 1], E)$$

is an isomorphism of topological vector spaces; the inverse map is

$$\Sigma^{-1}: L^1([0, 1], E) \rightarrow \bigoplus_{j \in J} L^1([0, 1], E_j), \quad ([\gamma])_{j \in J} \mapsto ([\gamma_j])_{j \in J}.$$

Consider the map

$$\Phi: \bigoplus_{j \in J} C([0, 1], H_j) \rightarrow C([0, 1], G)$$

taking $(\gamma_j)_{j \in J}$ to the function with components γ_j . Then Φ is a group homomorphism and smooth, because Φ takes the open set $\bigoplus_{j \in J} C([0, 1], U_j)$ into $C([0, 1], U)$ and

$$C([0, 1], \phi) \circ \Phi \circ \bigoplus_{j \in J} C([0, 1], \phi_j)^{-1}$$

is the restriction of the continuous linear summation map

$$\bigoplus_{j \in J} C([0, 1], E_j) \rightarrow C([0, 1], E)$$

and hence smooth. Since each of the maps $\text{Evol}_{H_j}: L^1([0, 1], E_j) \rightarrow C([0, 1], H_j)$ is smooth, also

$$\bigoplus_{j \in J} \text{Evol}_{H_j}: \bigoplus_{j \in J} L^1([0, 1], E_j) \rightarrow \bigoplus_{j \in J} C([0, 1], H_j)$$

is smooth (see [24, Proposition 7.1]). By (57) and (58), we have

$$\text{Evol}_G = \Phi \circ \left(\bigoplus_{j \in J} \text{Evol}_{H_j} \right) \circ \Sigma^{-1}.$$

Hence $\text{Evol}_G: L^1([0, 1], E) \rightarrow C([0, 1], G)$ is smooth, being a composition of smooth maps. As a consequence, $\text{Evol}: L^1([0, 1], E) \rightarrow AC_{L^1}([0, 1], G)$ is smooth (see Proposition 5.20) and thus G is L^1 -regular. \square

Theorem D from the introduction now follows as a corollary.

Proof of Theorem D. Let $(M_j)_{j \in J}$ be a locally finite family of compact submanifolds $M_j \subseteq M$ with boundary, of full dimension, such that the interiors M_j^0 cover M . Then

$$G := \bigoplus_{j \in J} C^k(M_j, H)$$

is L^1 -regular (see Remark 7.12 and Proposition 8.2). Let E be the modelling space of H . Then

$$S := \{\gamma = (\gamma_j)_{j \in J} \in G: (\forall j_1, j_2 \in J)(\forall x \in M_{j_1} \cap M_{j_2}) \gamma_{j_1}(x) = \gamma_{j_2}(x)\}$$

is a Lie subgroup of G modelled on

$$F := \{(\gamma_j)_{j \in J} \in \bigoplus_{j \in J} C^k(M_j, E): (\forall j_1, j_2 \in J)(\forall x \in M_{j_1} \cap M_{j_2}) \gamma_{j_1}(x) = \gamma_{j_2}(x)\},$$

which is a complemented vector subspace of $\bigoplus_{j \in J} C^k(M_j, E)$ (cf. Remark A.16). To see this, let $\phi: U \rightarrow V \subseteq E$ be a chart for H defined on an open identity neighbourhood $U \subseteq H$ such that $U = U^{-1}$. Then

$$\psi := \bigoplus_{j \in J} C^k(M_j, \phi): \bigoplus_{j \in J} C^k(M_j, U) \rightarrow \bigoplus_{j \in J} C^k(M_j, V)$$

is a chart for G such that

$$\psi \left(S \cap \bigoplus_{j \in J} C^k(M_j, U) \right) = F \cap \bigoplus_{j \in J} C^k(M_j, V).$$

As the maps

$$G \rightarrow H, \quad \gamma \mapsto \gamma_j(x)$$

are smooth group homomorphisms for all $j \in J$ and $x \in M_j$, all hypotheses of Proposition 5.27 are satisfied and thus S is L^1 -regular. The map

$$\Psi: C_c^k(M, E) \rightarrow F, \quad \gamma \mapsto (\gamma|_{M_j})_{j \in J}$$

is an isomorphism of topological vector spaces. Recall that $C_c^k(M, \phi)$ is a chart for $C_c^k(M, H)$. Moreover, ψ restricts to a chart

$$\psi_S: S \cap \bigoplus_{j \in J} C^k(M_j, U) \rightarrow F \cap \bigoplus_{j \in J} C^k(M_j, V).$$

It remains to observe that the map

$$\Theta: C_c^k(M, H) \rightarrow S, \quad \gamma \mapsto (\gamma|_{M_j})_{j \in J}$$

is an isomorphism of groups such that

$$\Theta(C_c^k(M, U)) = S \cap \bigoplus_{j \in J} C^k(M_j, U)$$

and $\psi_S \circ \Theta \circ C_c^k(M, \phi)^{-1}$ is the restriction of Ψ to a C^∞ -diffeomorphism

$$C_c^k(M, V) \rightarrow F \cap \bigoplus_{j \in J} C^k(M_j, E).$$

Hence Θ is an isomorphism of Lie groups and thus also the Lie group $C_c^k(M, H)$ is L^1 -regular (being isomorphic to the L^1 -regular Lie group S). \square

We record a second corollary to Proposition 8.2.

Corollary 8.3 *Let M be a finite-dimensional paracompact smooth manifold, H be a Banach-Lie group and $\pi: P \rightarrow M$ be a smooth principal bundle with structure group H . Then $\text{Gau}_c(P)$ is L^1 -regular.*

Proof. Let $(M_j)_{j \in J}$ be a locally finite family of compact submanifolds $M_j \subseteq M$ with boundary, of full dimension, such that the interiors M_j^0 cover M for $j \in J$ and there is a smooth section $\sigma_j: M_j \rightarrow P$ for π , for each $j \in J$. For all $i, j \in J$, there is a unique map $k_{i,j}: M_i \cap M_j \rightarrow H$ such that

$$\sigma_i(x)k_{i,j}(x) = \sigma_j(x) \quad \text{for all } x \in M_i \cap M_j.$$

Then $\text{Gau}_c(P)$ can be identified with the Lie subgroup S of

$$G := \bigoplus_{j \in J} C^\infty(M_j, H)$$

consisting of all $\gamma = (\gamma_j)_{j \in J} \in G$ such that

$$(\forall i, j \in J)(\forall x \in M_i \cap M_j) \gamma_i(x) = k_{i,j}(x)\gamma_j(x)k_{j,i}(x)$$

(see [65] if M is σ -compact).²⁶ Each of the Lie groups $C^\infty(M_j, H)$ is L^1 -regular (see Remark 7.12), whence also $G = \bigoplus_{j \in J} C^\infty(M_j, H)$ is L^1 -regular (by Proposition 8.2). For $i, j \in J$ and $x \in M_i \cap M_j$, the mappings

$$\alpha_{i,j,x}: G \rightarrow H, \quad \gamma \mapsto \gamma_i(x)$$

and

$$\beta_{i,j,k}: G \rightarrow H, \quad \gamma \mapsto k_{i,j}(x)\gamma_j(x)k_{j,i}(x)$$

are smooth group homomorphisms. Since

$$S = \{\gamma \in G: (\forall i, j \in \{1, \dots, m\})(\forall x \in M_i \cap M_j) \alpha_{i,j,x}(\gamma) = \beta_{i,j,x}(\gamma)\},$$

we deduce with Proposition 5.27 that S (and hence also $\text{Gau}_c(P)$) is L^1 -regular, using that modelling space is complemented (cf. Lemma A.15). \square

See [34, Lemma 7.1] for the following tool:

Lemma 8.4 *Let M and N be C^1 -manifolds modeled on locally convex spaces and $f: M \rightarrow N$ be a map. Then f is C^1 if and only if there exists a continuous map $\omega: TM \rightarrow TN$ with the following properties:*

- (a) $\omega(T_x M) \subseteq T_{f(x)} N$ for each $x \in M$;

²⁶We can use this identification to define the Lie group structure on $\text{Gau}_c(P)$.

- (b) If $\varepsilon > 0$ and $\gamma:]-\varepsilon, \varepsilon[\rightarrow M$ is a C^1 -map, then $f \circ \gamma$ is C^1 with $(f \circ \gamma)'(0) = \omega(\gamma'(0))$.

In this case, $Tf = \omega$. If M is an open subset of a locally convex space X , it suffices to take paths of the form $\gamma(s) = x + sy$ in (b), for $x \in M$ and $y \in X$. \square

The next lemma motivates the definition of ω in the proof of Proposition 8.6 (and is used in the proof of Corollary 8.7).

Lemma 8.5 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let G be a Lie group modelled on such a space. If G is \mathcal{E} -regular and $[\gamma], \eta \in \mathcal{E}([0, 1], \mathfrak{g})$, then*

$$\theta: \mathbb{R} \rightarrow AC_{\mathcal{E}}([0, 1], G), \quad s \mapsto \text{Evol}(\eta + s[\gamma])$$

is a smooth curve in $AC_{\mathcal{E}}([0, 1], G)$ with right logarithmic derivative

$$\delta^r(\theta)(s) = I_{\mathfrak{g}}([\text{Ad}(\text{Evol}(\eta + s[\gamma])).\gamma]) = I_{\mathfrak{g}}((\eta + s[\gamma] + [\gamma]) \odot (\eta + s[\gamma])^{-1}) \quad (59)$$

for $s \in \mathbb{R}$, where $I_{\mathfrak{g}}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow AC_{\mathcal{E}}([0, 1], \mathfrak{g})$, $[\zeta] \mapsto (t \mapsto \int_0^t \zeta(\tau) d\tau)$.

Proof. Since $\text{Evol}: (\mathcal{E}([0, 1], \mathfrak{g}), \odot) \rightarrow AC_{\mathcal{E}}([0, 1], G)$ is a smooth homomorphism between Lie groups, we have

$$\begin{aligned} (\delta^r \theta)(s) &= L(\text{Evol})\delta^r(s \mapsto \eta + s[\gamma]) \\ &= I_{\mathfrak{g}}d\rho_{(\eta + s[\gamma])^{-1}}(\eta + s[\gamma], [\gamma]) \\ &= I_{\mathfrak{g}}([\text{Ad}(\text{Evol}(\eta + s[\gamma])).\gamma]). \end{aligned}$$

This establishes the first equality in (59) and the second equality is merely a rewriting with the help of (32) in Lemma 5.12. \square

Proposition 8.6 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let G be a \mathcal{E} -semiregular Lie group modelled on such a space, with Lie algebra \mathfrak{g} . Define*

$$I_{\mathfrak{g}}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow C([0, 1], \mathfrak{g}), \quad \zeta \mapsto \left(t \mapsto \int_0^t \zeta(\tau) d\tau \right).$$

Assume that

- (a) $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow C([0, 1], G)$ is continuous;
- (b) The map $\mathbb{R} \rightarrow C([0, 1], G)$, $s \mapsto \text{Evol}(\eta + s[\gamma])$ is C^1 for all $[\gamma], \eta \in \mathcal{E}([0, 1], \mathfrak{g})$; and
- (c) $\left. \frac{d}{ds} \right|_{s=0} \text{Evol}(\eta + s[\gamma]) = I_{\mathfrak{g}}([\text{Ad}(\text{Evol}(\eta))\gamma]) \cdot \text{Evol}(\eta)$, where the dot means multiplication in the tangent Lie group $T(C([0, 1], G))$ and we identify $T_1 C([0, 1], G)$ with $C([0, 1], \mathfrak{g})$.

Then G is \mathcal{E} -regular.

Proof. Since Evol is continuous by assumption, $(\mathcal{E}([0, 1], \mathfrak{g}), \odot)$ is a topological group by Proposition 5.38 (e). Thus $\omega: T\mathcal{E}([0, 1], \mathfrak{g}) \rightarrow T(C([0, 1], G))$,

$$\begin{aligned} \omega(\eta, [\gamma]) &:= I_{\mathfrak{g}}(d\rho_{\eta^{-1}}(\eta, [\gamma])) \cdot \text{Evol}(\eta) = I_{\mathfrak{g}}([\text{Ad}(\text{Evol}(\eta))\gamma]) \cdot \text{Evol}(\eta) \\ &= I_{\mathfrak{g}}([\gamma] + \eta) \odot \eta^{-1} \cdot \text{Evol}(\eta) \end{aligned}$$

is continuous. Moreover, ω takes $T_{\eta}\mathcal{E}([0, 1], \mathfrak{g}) = \{\eta\} \times \mathcal{E}([0, 1], \mathfrak{g})$ inside $T_{\text{Evol}(\eta)}(C([0, 1], G))$. Hence, by Lemma 8.4, Evol will be C^1 if we can show that, for all $[\gamma], \eta \in \mathcal{E}([0, 1], \mathfrak{g})$, the curve

$$\xi: \mathbb{R} \rightarrow C([0, 1], G), \quad \xi(s) := \text{Evol}(\eta + s[\gamma])$$

is C^1 and satisfies

$$\xi'(0) = \omega(\eta, [\gamma]). \tag{60}$$

But this is the case by hypotheses (b) and (c). \square

Corollary 8.7 *Let \mathcal{E} be a bifunctor on Fréchet spaces (resp., on sequentially complete (FEP)-spaces, resp., on integral complete locally convex spaces) which satisfies the locality axiom, the pushforward axioms, and such that smooth functions act smoothly on $AC_{\mathcal{E}}$. Let (J, \leq) be a directed set,*

$$((G_j)_{j \in J}, (\alpha_{i,j})_{i \geq j})$$

be a direct system of \mathcal{E} -semiregular Lie groups G_j modelled on spaces as just described and smooth homomorphisms $\alpha_{i,j}: G_j \rightarrow G_i$. Let G be a Lie group modelled on a space as just described and $\alpha_j: G_j \rightarrow G$ be smooth homomorphisms for $j \in J$ such that $\alpha_i \circ \alpha_{i,j} = \alpha_j$ for all $i, j \in J$ such that $i \geq j$. Let $\mathfrak{g} := L(G)$ and $\mathfrak{g}_j := L(G_j)$. Assume that each $[\gamma] \in \mathcal{E}([0, 1], \mathfrak{g})$ is of the form $[L(\alpha_j) \circ \zeta]$ for some $j \in J$ and $[\zeta] \in \mathcal{E}([0, 1], \mathfrak{g}_j)$. Then G is \mathcal{E} -semiregular. If, moreover, $\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow C([0, 1], G)$ is continuous at 0 and each G_j is \mathcal{E} -regular, then G is \mathcal{E} -regular.

Proof. If $[\gamma] \in \mathcal{E}([0, 1], \mathfrak{g})$ and $[\zeta] \in \mathcal{E}([0, 1], \mathfrak{g}_j)$ such that $[\gamma] = [L(\alpha_j) \circ \zeta]$, then $\alpha_j \circ \text{Evol}_{G_j}([\zeta])$ is the left evolution of $[\gamma]$. Hence G is \mathcal{E} -semiregular. Now assume that

$$\text{Evol}: \mathcal{E}([0, 1], \mathfrak{g}) \rightarrow C([0, 1], G)$$

is continuous at 0 and hence continuous, by Proposition 5.38 (e). If $[\gamma], [\eta] \in \mathcal{E}([0, 1], \mathfrak{g})$, then there are $i, j \in J$ and $[\zeta] \in \mathcal{E}([0, 1], \mathfrak{g}_i)$, $[\xi] \in \mathcal{E}([0, 1], \mathfrak{g}_j)$ such that $[\gamma] = [L(\alpha_i) \circ \zeta]$ and $[\eta] = [L(\alpha_j) \circ \xi]$. Since J is directed, there is $\ell \in J$ such that $\ell \geq i, j$. Let $\bar{\zeta} := L(\alpha_{\ell,i}) \circ \zeta$ and $\bar{\xi} := L(\alpha_{\ell,j}) \circ \xi$. Then $[\bar{\zeta}], [\bar{\xi}] \in \mathcal{E}([0, 1], \mathfrak{g}_\ell)$ and $[L(\alpha_\ell) \circ \bar{\zeta}] = [L(\alpha_\ell) \circ L(\alpha_{\ell,i}) \circ \zeta] = [L(\alpha_\ell \circ \alpha_{\ell,i}) \circ \zeta] = [L(\alpha_i) \circ \zeta] = [\gamma]$; likewise, $[L(\alpha_\ell) \circ \bar{\xi}] = [\eta]$. Note that

$$\theta: \mathbb{R} \rightarrow AC_{\mathcal{E}}([0, 1], G_\ell), \quad \theta(s) := \text{Evol}_{G_\ell}([\bar{\zeta}] + s[\bar{\xi}])$$

is a C^1 -curve in $AC_{\mathcal{E}}([0, 1], G_\ell)$ with

$$\theta'(s) = I_{\mathfrak{g}_\ell}([\text{Ad}(\text{Evol}_{G_\ell}([\bar{\zeta}])\bar{\xi})]). \text{Evol}_{G_\ell}([\bar{\zeta}]),$$

by Lemma 8.5. Hence $\alpha_\ell \circ \theta: \mathbb{R} \rightarrow C([0, 1], G)$, $s \mapsto \text{Evol}([\zeta] + s[\xi])$ is a C^1 -curve and

$$\begin{aligned} \left. \frac{d}{ds} \right|_{s=0} \text{Evol}([\zeta] + s[\xi]) &= T(\alpha_\ell)(\theta'(0)) \\ &= L(\alpha_\ell)(I_{\mathfrak{g}_\ell}([\text{Ad}(\text{Evol}_{G_\ell}([\bar{\zeta}])\bar{\xi})])). \alpha_\ell(\text{Evol}_{G_\ell}([\bar{\zeta}])) \\ &= I_{\mathfrak{g}}([\text{Ad}(\text{Evol}([\zeta])\xi)]. \text{Evol}([\zeta]). \end{aligned}$$

Now apply Proposition 8.6. □

To get ahead, we need a better understanding of Lebesgue spaces with values in a locally convex direct limit. Our next result was stimulated by the following fact [53]:

Mujica's Theorem. *If X is a compact topological space and a locally convex space E is the direct limit of an ascending sequence $E_1 \subseteq E_2 \subseteq \cdots$ of locally convex spaces (with continuous inclusion maps), then the natural map*

$$\Phi: \lim_{\longrightarrow} C(X, E_n) \rightarrow C(X, E)$$

*induced by the inclusion maps is a topological embedding.*²⁷

²⁷If the locally convex direct limit topology is used on the left-hand side.

Hence Φ is an isomorphism of topological vector spaces if $E = \bigcup_{n \in \mathbb{N}} E_n$ is compact regular. This important special case was first obtained by Schmets [64] (and, earlier, by Mujica in the special case of (LB)-spaces). Our analogue of Mujica's Theorem for Lebesgue spaces reads as follows.

Proposition 8.8 *Let $E_1 \subseteq E_2 \subseteq \dots$ be an ascending sequence of locally convex spaces such that the inclusion maps $E_n \rightarrow E_{n+1}$ are continuous linear. Endow $E = \bigcup_{n \in \mathbb{N}} E_n$ with the locally convex direct limit topology and assume the latter is Hausdorff. Let (X, Σ, μ) be a measure space. Then we have:*

(a) *The injective continuous linear map*

$$\Phi: \bigcup_{n \in \mathbb{N}} L_{rc}^\infty(X, \mu, E_n) \rightarrow L_{rc}^\infty(X, \mu, E)$$

induced by the inclusion maps is a topological embedding with respect to the locally convex direct limit topology on the union on the left. If the direct limit $E = \varinjlim E_n$ is compact regular,²⁸ then Φ is an isomorphism of topological vector spaces.

(b) *If E and each E_n has the (FEP), then for each $p \in [1, \infty[$ the injective continuous linear map*

$$\Phi: \bigcup_{n \in \mathbb{N}} L^p(X, \mu, E_n) \rightarrow L^p(X, \mu, E)$$

induced by the inclusion maps is a topological embedding with respect to the locally convex direct limit topology on the union on the left. If $E = \varinjlim E_n$ is a strict (LF)-space, then Φ is an isomorphism of topological vector spaces.

Proof. (a) By construction, Φ is injective, continuous and linear. As a consequence, the locally convex direct limit topology on the left is Hausdorff. To see that Φ is open onto its image, it suffices to show that $\Phi(\overline{W})$ is a zero-neighbourhood in $\text{im}(\Phi)$ for W in a basis of 0-neighbourhoods in the locally convex direct limit $\bigcup_{n \in \mathbb{N}} L_{rc}^\infty(X, \mu, E_n)$ (as every topological vector space is a regular topological space, we still have a basis if we pass to the closures \overline{W}). By Lemma 1.37, we may assume that

$$W = \left\{ \sum_{n=1}^{\infty} [\gamma_n] : ([\gamma_n])_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L_{rc}^\infty(X, \mu, E_n) : \sup\{\|\gamma_n\|_{\mathcal{L}^\infty, q_n} : n \in \mathbb{N}\} < 1 \right\}$$

²⁸It suffices that every compact metrizable subset of E is a compact subset of some E_n .

for some sequence $(q_n)_{n \in \mathbb{N}}$ of continuous seminorms $q_n: E_n \rightarrow [0, \infty[$. By the same lemma,

$$V := \left\{ \sum_{n=1}^{\infty} v_n : (v_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} E_n : \sup\{q_n(v_n) : n \in \mathbb{N}\} < 1 \right\}$$

is a 0-neighbourhood in E . Hence $L_{rc}^{\infty}(X, \mu, V^0)$ is an open 0-neighbourhood in $L_{rc}^{\infty}(X, \mu, E)$ and it suffices to prove that

$$\text{im}(\Phi) \cap L_{rc}^{\infty}(X, \mu, V^0) \subseteq \Phi(\overline{W}).$$

To this end, let $[\gamma] \in \text{im}(\Phi) \cap L_{rc}^{\infty}(X, \mu, V^0)$. We may assume that $\gamma \in \mathcal{L}_{rc}^{\infty}(X, \mu, E_N)$ for some n and (possibly after changing it to 0 on a set of measure zero) that $\overline{\gamma(X)} \subseteq V^0$. Thus $\gamma(X) \subseteq V$. There is a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of finitely-valued measurable functions $\gamma_n: X \rightarrow E_N$ with $\gamma_n(X) \subseteq \gamma(X) \subseteq V$ such that $\gamma_n(x) \rightarrow \gamma(x)$ in E_N , uniformly in $x \in X$. Hence $[\gamma_n] \rightarrow [\gamma]$ in $L_{rc}^{\infty}(X, \mu, E_N)$. If we can show that

$$[\gamma_n] \in W \text{ for each } n \in \mathbb{N},$$

then $[\gamma] \in \overline{W}$ and hence $[\gamma] = \Phi([\gamma]) \in \Phi(\overline{W})$. Fix $n \in \mathbb{N}$. We can write

$$\gamma_n = \sum_{i=1}^m c_i \mathbf{1}_{A_i}$$

with some $m \in \mathbb{N}$, $c_i \in E_N$ for $i \in \{1, \dots, m\}$ and disjoint, non-empty measurable sets $A_i \subseteq X$. Since $c_i \in V$, we can find $k \in \mathbb{N}$ such that

$$c_i = \sum_{j=1}^k v_{i,j}$$

with $v_{i,j} \in E_j$ for $j \in \{1, \dots, k\}$ and

$$\max\{q_j(v_{i,j}) : j = 1, \dots, k\} < 1.$$

Then $\gamma_n = \sum_{i=1}^m \sum_{j=1}^k v_{i,j} \mathbf{1}_{A_i} = \sum_{j=1}^k \eta_j$ with $\eta_j := \sum_{i=1}^m v_{i,j} \mathbf{1}_{A_i}$. We have $\eta_j \in \mathcal{L}_{rc}^{\infty}(X, \mu, E_j)$ and

$$\|\eta_j\|_{\mathcal{L}^{\infty, q_j}} = \max\{q_j(v_{i,j}) : i = 1, \dots, m\} < 1$$

(using that the sets A_i are disjoint). As a consequence, $[\gamma_n] = \sum_{j=1}^k [\eta_j] \in W$. This completes the proof that Φ is open onto its image. If E is compact regular, then Φ is surjective and hence (being also a linear topological embedding) an isomorphism of topological vector spaces.

(b) Again, Φ is continuous, linear and injective. To see that Φ is open onto its image, it suffices to show that $\Phi(\overline{W})$ is a zero-neighbourhood in $\text{im}(\Phi)$ for W in a basis of 0-neighbourhoods in the locally convex direct limit $\bigcup_{n \in \mathbb{N}} L_{rc}^\infty(X, \mu, E_n)$. By Lemma 1.37, we may assume that

$$W = \left\{ \sum_{n=1}^{\infty} [\gamma_n] : ([\gamma_n])_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L^p(X, \mu, E_n) : \left(\sum_{n=1}^{\infty} (\|\gamma_n\|_{\mathcal{L}^p, q_n})^p \right)^{1/p} < 1 \right\}$$

for some sequence $(q_n)_{n \in \mathbb{N}}$ of continuous seminorms $q_n : E_n \rightarrow [0, \infty[$. Using (as in the proof of Lemma 1.37) that E is a quotient of $\bigoplus_{n \in \mathbb{N}} E_n$, we deduce from Lemma 1.36 that

$$q(x) := \inf \left\{ \|(q_n(x))_{n \in \mathbb{N}}\|_{\ell^p} : (x_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} E_n \text{ with } x = \sum_{n=1}^{\infty} x_n \right\}$$

defines a continuous seminorm $q : E \rightarrow [0, \infty[$. It therefore suffices to prove that

$$\text{im}(\Phi) \cap \{[\gamma] \in L^p(X, \mu, E) : \|\gamma\|_{\mathcal{L}^p, q} < 1\} \subseteq \Phi(\overline{W}).$$

To this end, let $[\gamma] \in \text{im}(\Phi)$ such that $\|\gamma\|_{\mathcal{L}^p, q} < 1$. We may assume that $\gamma \in \mathcal{L}^p(X, \mu, E_N)$ for some n (possibly after changing γ to 0 on a set of measure zero). There is a net $(\gamma_\alpha)_{\alpha \in A}$ of finitely-valued measurable functions $\gamma_\alpha : X \rightarrow E_N$ with $\gamma_\alpha(X) \subseteq \gamma(X) \cup \{0\}$ and $\mu(\gamma_\alpha^{-1}(E_N \setminus \{0\})) < \infty$ such that $\gamma_\alpha \rightarrow \gamma$ in $\mathcal{L}^p(X, \mu, E_N)$ (see Lemma 1.44). If we can show that

$$[\gamma_\alpha] \in W \text{ for each } \alpha,$$

then $[\gamma] \in \overline{W}$ and hence $[\gamma] = \Phi([\gamma]) \in \Phi(\overline{W})$. Fix α . We can write

$$\gamma_\alpha = \sum_{i=1}^m c_i \mathbf{1}_{A_i}$$

with some $m \in \mathbb{N}$, $c_i \in E_N$ for $i \in \{1, \dots, m\}$ and disjoint, non-empty measurable sets $A_i \subseteq X$. Then

$$\|\gamma_\alpha\|_{\mathcal{L}^p, q} = \left(\sum_{i=1}^m (q(c_i))^p \mu(A_i) \right)^{1/p} < 1$$

and we can find $k \in \mathbb{N}$ such that

$$c_i = \sum_{j=1}^k v_{i,j}$$

with $v_{i,j} \in E_j$ for $j \in \{1, \dots, k\}$ and

$$\left(\sum_{i=1}^m \sum_{j=1}^k (q_j(v_{i,j}))^p \mu(A_i) \right)^{1/p} < 1. \quad (61)$$

Then $\gamma_\alpha = \sum_{i=1}^m \sum_{j=1}^k v_{i,j} \mathbf{1}_{A_i} = \sum_{j=1}^k \eta_j$ with $\eta_j := \sum_{i=1}^m v_{i,j} \mathbf{1}_{A_i}$. We have $\eta_j \in \mathcal{L}^p(X, \mu, E_j)$ and

$$\|\eta_j\|_{\mathcal{L}^p, q_j} = \left(\sum_{i=1}^m (q_j(v_{i,j}))^p \mu(A_i) \right)^{1/p}$$

(using that the sets A_i are disjoint). As a consequence, $[\gamma_\alpha] = \sum_{j=1}^k [\eta_j]$ with

$$\left(\sum_{j=1}^k (\|\eta_j\|_{\mathcal{L}^p, q_n})^p \right)^{1/p} = \left(\sum_{j=1}^k \sum_{i=1}^m (q_j(v_{i,j}))^p \mu(A_i) \right)^{1/p} < 1,$$

by (61). Thus $\eta_j \in W$, completing the proof that Φ is open onto its image. If $E = \bigcup_{n \in \mathbb{N}} E_n$ is a strict (LF)-space, let us show that Φ is surjective. We claim that for each $\gamma \in \mathcal{L}^p(X, \mu, E)$, there exists $N \in \mathbb{N}$ such that $\mu(\gamma^{-1}(E \setminus E_N)) = 0$. If this is true, then $[\gamma] = [\gamma \mathbf{1}_B] \in L^p(X, \mu, E_N)$ with $B := \gamma^{-1}(E_N)$ and the proof is complete. To prove the claim, we assume it is wrong and deduce a contradiction. Thus, suppose there is an element $\gamma \in \mathcal{L}^p(X, \mu, E)$ such that $\mu(\gamma^{-1}(E \setminus E_N)) > 0$ for each $N \in \mathbb{N}$. Since

$$\mu(\gamma^{-1}(E \setminus E_N)) = \sum_{n=N}^{\infty} \mu(\gamma^{-1}(E_{n+1} \setminus E_n)),$$

there exists $n \geq N$ such that $\mu(\gamma^{-1}(E_{n+1} \setminus E_n)) > 0$. Thus, using a simple induction, we find a sequence

$$n_1 < n_2 < \dots$$

of positive integers such that $\mu(\gamma^{-1}(E_{n_{j+1}} \setminus E_{n_j})) > 0$ for each $j \in \mathbb{N}$. Let $\rho_j: E \rightarrow E/E_{n_j}$ be the canonical quotient map. Since $\rho_j \circ \gamma \in \mathcal{L}^p(X, \mu, E/E_{n_j})$ does not vanish almost everywhere, we find a continuous seminorm Q_j on E/E_{n_j} such that $\rho_j \circ \gamma \|_{\mathcal{L}^p, Q_j} > 0$. Then $q_j := Q_j \circ \rho_j$ is a continuous seminorm on E which vanishes on E_{n_j} . After replacing Q_j with a large multiple, we may assume that

$$\|\gamma\|_{\mathcal{L}^p, q_j} = \|\rho_j \circ \gamma\|_{\mathcal{L}^p, Q_j} \geq j.$$

Now $q := \sum_{j=1}^{\infty} q_j$ coincides with the finite sum $\sum_{i=1}^{j-1} q_j$ on E_{n_j} (as q_i vanishes on E_{n_i} and hence on E_{n_j} for each $i \geq j$). Hence $q|_{E_{n_j}}$ is a continuous seminorm for each j and hence q is a continuous seminorm on $E = \varinjlim E_{n_j}$. Since

$$\|\gamma\|_{\mathcal{L}^p, q} \geq \|\gamma\|_{\mathcal{L}^p, q_j} \geq j$$

for each $j \in \mathbb{N}$, we cannot have $\|\gamma\|_{\mathcal{L}^p, q} < \infty$, contradicting the hypothesis that $\gamma \in \mathcal{L}^p(X, \mu, E)$. \square

Remark 8.9 A result for L^p -spaces very similar to Proposition 8.8 was already obtained by Mayoral et al. [8], using a different concept of vector-valued L^p -space which makes sense for μ a Radon measure on a σ -compact locally compact space X (defined on a σ -algebra Σ containing the Borel σ -algebra such that (X, Σ, μ) is a complete measure space). In the cited paper, a mapping $\gamma: X \rightarrow E$ to a locally convex space E is called μ -measurable²⁹ if there is a sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets $K_n \subseteq X$ such that

$$f|_{K_n}: K_n \rightarrow E$$

is continuous and

$$\mu\left(X \setminus \bigcup_{n \in \mathbb{N}} K_n\right) = 0;$$

$L^p(X, \mu, E)$ (denoted $L^p(\{E\})$ there) is defined as the space of equivalence classes of μ -measurable mappings $\gamma: X \rightarrow E$ such that $q \circ \gamma \in \mathcal{L}^p(X, \mu)$ for all continuous seminorms q on E . The special case of ℓ^p -spaces was discussed earlier in [18]. For related results concerning ℓ^1 , compare also [49].

Our next main goal is the following result:

²⁹This property is also known as *Lusin-measurability*.

Proposition 8.10 *Let G be a Lie group whose Lie algebra $\mathfrak{g} := L(G)$ is an (LB)-space. Assume that there exists a projective system*

$$((G_n)_{n \in \mathbb{N}}, (\psi_{n,m})_{n \geq m})$$

of Banach-Lie groups G_n with Lie algebras $\mathfrak{g}_n := L(G_n)$ and smooth homomorphisms $\psi_{n,m}: G_m \rightarrow G_n$ such that $L(\psi_{n,m})$ is injective, and smooth homomorphisms $\psi_n: G_n \rightarrow G$ such that $\psi_n \circ \psi_{n,m} = \psi_m$ for all positive integers $n \geq m$ and

$$\mathfrak{g} = \varinjlim \mathfrak{g}_n$$

with the limit maps $L(\psi_n)$ and bonding maps $L(\psi_{n,m})$. Then $L(\psi_n)$ is injective for all n , enabling $x \in \mathfrak{g}_n$ to be identified with $L(\psi_n)(x) \in \mathfrak{g}$. Now $\mathfrak{g}_m \subseteq \mathfrak{g}_n$ if $m \leq n$ and $L(\psi_{n,m}): \mathfrak{g}_m \rightarrow \mathfrak{g}_n$ becomes the inclusion map. We show:

- (a) *If $\mathfrak{g} = \bigcup_{n \in \mathbb{N}} \mathfrak{g}_n$ is compact regular, then G is L_{rc}^∞ -regular.*
- (b) *If $\mathfrak{g} = \bigcup_{n \in \mathbb{N}} \mathfrak{g}_n$ is a strict direct limit, then \mathfrak{g} is an (FEP)-space and G is L^1 -regular.*

Remark 8.11 More generally, G as in Proposition 8.10 is L^1 -regular whenever \mathfrak{g} is an (FEP)-space and the natural map

$$\varinjlim L^1([0, 1], \mathfrak{g}_n) \rightarrow L^1([0, 1], \mathfrak{g})$$

induced by the maps $L^1([0, 1], L(\psi_n))$ is surjective (only these properties are used in the proof of (b)).

The following special case is easier to remember. Here $\mathfrak{g} := L(G)$ and $\mathfrak{g}_n := L(G_n)$, as usual.

Corollary 8.12 *Let G be a Lie group modelled on an (LB)-space and $(G_n)_{n \in \mathbb{N}}$ be a sequence of Banach-Lie groups such that*

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G$$

(with each inclusion a smooth group homomorphism) and $\mathfrak{g} = \varinjlim \mathfrak{g}_n$.

- (a) *If $\varinjlim \mathfrak{g}_n$ is compactly regular, then G is L_{rc}^∞ -regular.*

(b) If $\lim_{\rightarrow} \mathfrak{g}_n$ is a strict (LB)-space, then G is L^1 -regular. \square

Some auxiliary concepts and simple lemmas will help us to prove the preceding proposition. As usual, if E is a Banach space, write $(\mathcal{L}(E), \|\cdot\|_{op})$ for the Banach algebra of bounded linear operators $\alpha: E \rightarrow E$ and $\text{GL}(E) := \mathcal{L}(E)^\times$ for the group of invertible operators (which is an open subset of $\mathcal{L}(E)$ and a Banach-Lie group).

Definition 8.13 Let $(E, \|\cdot\|)$ be a Banach space. We say that a non-empty set $M \subseteq \text{GL}(E)$ of invertible operators is *uniformly expanding* if

$$\sup\{\|\alpha^{-1}\|_{op} : \alpha \in M\} < \infty.$$

Lemma 8.14 If $(E, \|\cdot\|)$ is a Banach space and $M \subseteq \text{GL}(E)$ a uniformly expanding set of invertible operators, define

$$s := \sup\{\|\alpha^{-1}\|_{op} : \alpha \in M\} \in]0, \infty[.$$

Then

$$\alpha(B_r^E(0)) \supseteq B_{r/s}^E(0) \tag{62}$$

for all $\alpha \in M$ and $r > 0$.

Proof. For $\alpha \in M$ and $r > 0$, we have

$$\alpha^{-1}(B_{r/s}^E(0)) \subseteq B_{(r\|\alpha^{-1}\|_{op})/s}^E(0) \subseteq B_r^E(0).$$

Thus $B_r^E(0) \supseteq \alpha^{-1}(B_{r/s}^E(0))$. Applying α to both sides of this inclusion, (62) follows. \square

Definition 8.15 Let G be a Banach-Lie group. We say that a subset $M \subseteq G$ is *product-exponential* (or, in short, a (PE)-subset), if there exists $n \in \mathbb{N}$ and non-empty bounded subsets $B_1, \dots, B_n \subseteq L(G)$ such that

$$M \subseteq \exp_G(B_1) \exp_G(B_2) \cdots \exp_G(B_n).$$

Lemma 8.16 Let $(E, \|\cdot\|)$ be a Banach space and $M \subseteq \text{GL}(E)$ be a (PE)-subset. Then M is uniformly expanding.

Proof. Let $B_1, \dots, B_n \subseteq \mathcal{L}(E)$ be non-empty bounded subsets such that $M \subseteq \exp(B_1) \cdots \exp(B_n)$, where $\exp(\alpha) := \sum_{k=0}^{\infty} \frac{1}{k!} \alpha^k$ for $\alpha \in \mathcal{L}(E)$. Since \exp is continuous and $\exp(0) = \text{id}_E$, there is a 0-neighbourhood $V \subseteq \mathcal{L}(E)$ such that

$$(\forall \alpha \in V) \quad \|\exp(\alpha)\|_{op} \leq 2.$$

Since each of the sets $-B_1, \dots, -B_n$ is bounded, we find $m \in \mathbb{N}$ such that

$$(\forall j \in \{1, \dots, n\}) \quad -B_j \subseteq mV$$

and thus

$$\|\exp(-B_j)\|_{op} \subseteq \|\exp(mV)\|_{op} \subseteq [0, 2^m]$$

for all $j \in \{1, \dots, n\}$, exploiting that

$$\|\exp(m\alpha)\|_{op} = \|\exp(\alpha)^m\|_{op} \leq (\|\exp(\alpha)\|_{op})^m \leq 2^m$$

for each $\alpha \in V$. If $\alpha_j \in B_j$ for $j \in \{1, \dots, n\}$, then

$$\begin{aligned} & \|(\exp(\alpha_1) \exp(\alpha_2) \cdots \exp(\alpha_n))^{-1}\|_{op} \\ &= \|\exp(-\alpha_n) \cdots \exp(-\alpha_2) \exp(-\alpha_1)\|_{op} \\ &\leq \|\exp(-\alpha_n)\|_{op} \cdots \|\exp(-\alpha_2)\|_{op} \|\exp(-\alpha_1)\|_{op} \leq 2^m \end{aligned}$$

by the preceding, whence

$$\sup\{\|\alpha^{-1}\|_{op} : \alpha \in M\} \leq 2^m < \infty.$$

Thus M is uniformly expanding. \square

Lemma 8.17 *If $\psi: G \rightarrow H$ is a smooth homomorphism between Banach-Lie groups and M a (PE)-subset of G , then $\psi(M)$ is a (PE)-subset of H . In particular, $\text{Ad}_H(\psi(M))$ is a (PE)-subset of $\text{GL}(\mathfrak{h})$ with $\mathfrak{h} := L(H)$.*

Proof. If M is a (PE)-subset of G , then $M \subseteq \exp_G(B_1) \cdots \exp_G(B_n)$ for suitable $n \in \mathbb{N}$ and bounded sets $B_1, \dots, B_n \subseteq G$. Since $\psi \circ \exp_G = \exp_H \circ L(\psi)$, we then have $\psi(M) \subseteq \exp_H(L(\psi)(B_1)) \cdots \exp_H(L(\psi)(B_n))$. As $L(\psi)(B_j) \subseteq L(H)$ is a bounded set for each $j \in \{1, \dots, n\}$, we see that $\psi(M)$ is a (PE)-subset of H . Since $\text{Ad}_H: H \rightarrow \text{GL}(\mathfrak{h})$ and hence also $\text{Ad}_H \circ \psi: G \rightarrow \text{GL}(\mathfrak{h})$

is a smooth homomorphism between Banach-Lie groups, the final assertion is a special case of the first. \square

In the next lemma, we endow $C([0, 1], \mathcal{L}(E))$ with the supremum norm $\|\cdot\|_\infty$,

$$\|\gamma\|_\infty := \sup\{\|\gamma(t)\|_{op} : t \in [0, 1]\}$$

for $\gamma \in C([0, 1], \mathcal{L}(E))$.

Lemma 8.18 *Let \mathcal{E} be L^p with $p \in [1, \infty]$ or L_{rc}^∞ . Let $(E, \|\cdot\|)$ be a Banach space and*

$$m_\gamma(\eta)(t) := \gamma(t)(\eta(t))$$

for $\gamma \in C([0, 1], \mathcal{L}(E))$ and $[\eta] \in \mathcal{E}([0, 1], E)$. Then

$$\psi(\gamma)([\eta]) := [m_\gamma(\eta)] \in \mathcal{E}([0, 1], E).$$

Moreover, $\psi(\gamma) \in \mathcal{L}(\mathcal{E}([0, 1], E))$ and the map

$$\psi : C([0, 1], \mathcal{L}(E)) \rightarrow \mathcal{L}(\mathcal{E}([0, 1], E))$$

so obtained is a homomorphism of unital Banach algebras with $\|\psi\|_{op} \leq 1$. If $M \subseteq C([0, 1], \text{GL}(E))$ is a non-empty subset such that

$$\{\gamma(t) : \gamma \in M, t \in [0, 1]\}$$

is a uniformly expanding subset of $\text{GL}(E)$ (for example, a (PE)-subset), then $\psi(M)$ is a uniformly expanding subset of $\text{GL}(\mathcal{E}([0, 1], E))$.

Proof. The evaluation map $\beta : \mathcal{L}(E) \times E \rightarrow E$, $\beta(\alpha, x) := \alpha(x)$ is continuous bilinear, whence $m_\gamma([\eta]) := [\beta \circ (\gamma, \eta)] \in \mathcal{E}([0, 1], E)$ for all $\gamma \in C([0, 1], \mathcal{L}(E))$ and $[\eta] \in \mathcal{E}([0, 1], E)$ by the pushforward axiom (P2) (see Lemma 5.8). Since $\|\beta(\gamma(t), \eta(t))\| = \|\gamma(t)(\eta(t))\| \leq \|\gamma(t)\|_{op} \|\eta(t)\| \leq \|\gamma\|_\infty \|\eta(t)\|$ almost everywhere, we see that

$$\|\psi(\gamma)([\eta])\|_{\mathcal{E}} \leq \|\gamma\|_\infty \|\eta\|_{\mathcal{E}},$$

whence $\|\psi(\gamma)\|_{op} \leq \|\gamma\|_\infty$ and thus $\|\psi\|_{op} \leq 1$. Moreover, apparently ψ is linear, multiplicative and $\psi(1) = \text{id}_{\mathcal{E}([0, 1], E)}$. If M is as described in the lemma, then

$$s := \sup\{\|(\gamma(t))^{-1}\|_{op} : \gamma \in M, t \in [0, 1]\} < \infty.$$

Now

$$\|(\psi(\gamma))^{-1}\|_{op} = \|\psi(\gamma^{-1})\|_{op} \leq \|\gamma^{-1}\|_{\infty} = \sup\{\|(\gamma(t))^{-1}\| : t \in [0, 1]\} \leq s$$

for each $\gamma \in \psi(M)$ and thus

$$\sup\{\|(\psi(\gamma))^{-1}\|_{op} : \gamma \in M\} \leq s < \infty.$$

Thus $\psi(M)$ is uniformly expanding. \square

Proof of Proposition 8.10. (a) Since $\Phi: \varinjlim L_{rc}^{\infty}([0, 1], \mathfrak{g}_n) \rightarrow L_{rc}^{\infty}([0, 1], \mathfrak{g})$ is an isomorphism (and hence surjective) by Proposition 8.8, Corollary 8.7 shows that G is L_{rc}^{∞} -semiregular. Let $U \subseteq C([0, 1], G)$ be an identity neighbourhood. After shrinking U , we may assume that $U = C([0, 1], U_0)$ for some identity neighbourhood $U_0 \subseteq G$. Pick identity neighbourhoods $U_n \subseteq G$ for $n \in \mathbb{N}$ such that

$$U_n U_n \subseteq U_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Then

$$U_n U_{n-1} \cdots U_1 \subseteq U_0 \quad \text{for all } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, the exponential map $\exp_{G_n}: \mathfrak{g}_n \rightarrow G_n$ is a local diffeomorphism at 0, whence we find an open 0-neighbourhood $B_n \subseteq \mathfrak{g}_n$ such that $V_n := \exp_{G_n}(B_n)$ is an open identity neighbourhood in G_n and $\exp_{G_n}: B_n \rightarrow V_n$ a diffeomorphism. After shrinking B_n , we may assume that B_n is bounded and $V_n \subseteq U_n$; thus V_n is an open identity neighbourhood and a (PE)-set. Now $C([0, 1], V_n)$ is an open identity neighbourhood in $C([0, 1], G_n)$. Since G_n is L^1 -regular, there is an open 0-neighbourhood $P_n \subseteq L_{rc}^{\infty}([0, 1], \mathfrak{g}_n)$ such that $\text{Evol}_{G_n}(P_n) \subseteq C([0, 1], V_n)$. We claim that

$$P := \bigcup_{n \in \mathbb{N}} (P_n \odot P_{n-1} \cdots \odot P_2 \odot P_1)$$

is a 0-neighbourhood in $L_{rc}^{\infty}([0, 1], \mathfrak{g})$. If this is true, then

$$\begin{aligned} \text{Evol}(P_n \odot \cdots \odot P_1) &= \text{Evol}(P_n) \cdots \text{Evol}(P_2) \text{Evol}(P_1) \\ &= \text{Evol}_{G_n}(P_n) \cdots \text{Evol}_{G_2}(P_2) \text{Evol}_{G_1}(P_1) \\ &\subseteq C([0, 1], V_n) \cdots C([0, 1], V_2) C([0, 1], V_1) \\ &\subseteq C([0, 1], U_n) \cdots C([0, 1], U_2) C([0, 1], U_1) \\ &\subseteq C([0, 1], U_n \cdots U_2 U_1) \subseteq C([0, 1], U_0) = U \end{aligned}$$

entails

$$\text{Evol}(P) = \bigcup_{n \in \mathbb{N}} \text{Evol}(P_n \odot \cdots \odot P_1) \subseteq U.$$

Hence Evol is continuous at 0 and thus G is L_{rc}^∞ -regular, by Corollary 8.7. To establish the claim, it suffices to find a sequence $(Q_n)_{n \in \mathbb{N}}$ of open convex 0-neighbourhoods $Q_n \subseteq L^\infty([0, 1], \mathfrak{g}_n)$ such that, for all $n \in \mathbb{N}$,

$$Q_1 + \cdots + Q_n \subseteq P_n \odot \cdots \odot P_1.$$

Then $Q := \bigcup_{n \in \mathbb{N}} (Q_1 + \cdots + Q_n)$ is a 0-neighbourhood in the locally convex direct limit $L_{rc}^\infty([0, 1], \mathfrak{g}) = \varinjlim L_{rc}^\infty([0, 1], \mathfrak{g}_n)$ such that $Q \subseteq P$, and so P is a 0-neighbourhood as well. Let F_n be the Banach space $L_{rc}^\infty([0, 1], \mathfrak{g}_n)$; we may assume that $P_n = B_{r_n}^{F_n}(0)$ for some $r_n > 0$. Set $Q_1 := P_1$. Let $n > 1$. For $k \in \{1, \dots, n-1\}$ and $\gamma \in P_k$, we have

$$\begin{aligned} \text{Ad}_{G_n}(\text{Evol}_{G_n}(\gamma)(t))^{-1} &= \text{Ad}_{G_n}(\text{Evol}_{G_k}(\gamma)(t))^{-1} \subseteq \text{Ad}_{G_n}(V_k)^{-1} \\ &= \text{Ad}_{G_n}(\exp_{G_k}(-B_k)). \end{aligned}$$

Thus

$$\begin{aligned} &\text{Ad}_{G_n}(\text{Evol}_{G_n}(\gamma_{n-1} \odot \cdots \odot \gamma_1)(t))^{-1} \\ &= \text{Ad}_{G_n}((\text{Evol}_{G_n}(\gamma_{n-1}(t)) \cdots \text{Evol}_{G_n}(\gamma_1(t))))^{-1} \\ &= \text{Ad}_{G_n}(\text{Evol}_{G_n}(\gamma_1(t))^{-1}) \cdots \text{Ad}_{G_n}(\text{Evol}_{G_n}(\gamma_{n-1}(t))^{-1}) \\ &\subseteq \text{Ad}_{G_n}(\exp_{G_1}(-B_1)) \cdots \text{Ad}_{G_n}(\exp_{G_{n-1}}(-B_{n-1})), \end{aligned}$$

from which we deduce (with Lemma 8.17) that

$$\{\text{Ad}_{G_n}(\text{Evol}_{G_n}(\eta)(t))^{-1} : \eta \in P_{n-1} \odot \cdots \odot P_1, t \in [0, 1]\}$$

is a (PE)-subset of $\text{GL}(\mathfrak{g}_n)$ and hence uniformly expanding (by Lemma 8.16).

Thus

$$M_n := \{[\zeta] \mapsto [\text{Ad}_{G_n}(\text{Evol}_{G_n}(\eta))^{-1} \cdot \zeta] : \eta \in P_{n-1} \odot \cdots \odot P_1\}$$

is a uniformly expanding subset of $\text{GL}(L_{rc}^\infty([0, 1], \mathfrak{g}_n))$, by Lemma 8.18. As a consequence,

$$s_n := \sup\{\|\alpha^{-1}\|_{op} : \alpha \in M_n\} < \infty$$

and hence

$$\text{Ad}_{G_n}(\text{Evol}_{G_n}(\eta))^{-1} \cdot P_n = \text{Ad}_{G_n}(\text{Evol}_{G_n}(\eta))^{-1} \cdot B_{r_n}^{F_n}(0) \subseteq B_{r_n/s_n}^{F_n}(0) =: Q_n$$

for each $\eta \in P_{n-1} \odot \cdots \odot P_1$. Since $[\gamma] \odot \eta = [\text{Ad}_{G_n}(\text{Evol}(\eta)(t))^{-1}\gamma(t)] + \eta$ for all $[\gamma] \in P_n$ and $\eta \in P_{n-1} \odot \cdots \odot P_1$, we deduce that

$$P_1 \odot \eta \supseteq Q_n + \eta$$

for each $\eta \in P_{n-1} \odot \cdots \odot P_1$. Hence

$$P_n \odot \cdots \odot P_1 \supseteq Q_n + P_{n+1} \odot \cdots \odot P_1 \supseteq Q_n + Q_{n-1} + \cdots + Q_1,$$

which completes the proof of (a).

(b) Replace L_{rc}^∞ with L^1 in the proof of (a). \square

As a first consequence, we see that direct limits of finite-dimensional Lie groups [27] are L_{rc}^∞ -regular:

Proof of Theorem E. Since $L(G) = \varinjlim L(G_n)$ is a strict (LB)-space, Corollary 8.12 applies. \square

Another application are Lie groups of real analytic maps. We first consider groups of germs of Lie group-valued analytic maps (as in [16]):

Corollary 8.19 *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let M is a \mathbb{K} -analytic manifold modelled on a Fréchet space, $K \subseteq M$ a non-empty compact set and H a \mathbb{K} -analytic Banach-Lie group; if $\mathbb{K} = \mathbb{R}$, assume that the topological space underlying M is regular. Then $\text{Germ}_{\mathbb{K}}(K, M, H)$ is an L_{rc}^∞ -regular \mathbb{K} -analytic Lie group. If H is finite-dimensional, then $G := \text{Germ}_{\mathbb{K}}(K, M, H)$ has a \mathbb{K} -analytic evolution map*

$$\text{Evol}: L_{rc}^\infty([0, 1], L(G)) \rightarrow AC_{L_{rc}^\infty}([0, 1], G).$$

Proof. The case $\mathbb{K} = \mathbb{C}$: Let $U_1 \supseteq U_2 \supseteq \cdots$ be a basis of open neighbourhoods of K in M such that each connected component of U_n meets M . Let $G := \text{Germ}_{\mathbb{C}}(K, M, H)$, $\mathfrak{h} := L(H)$ and $\mathfrak{g} := \text{Germ}_{\mathbb{C}}(K, M, \mathfrak{h})$. Then $\mathfrak{g}_n := \text{Hol}_b(U_n, \mathfrak{h})$ with the supremum norm is a Banach-Lie algebra. Moreover, \mathfrak{g}_n can be identified with a vector subspace of \mathfrak{g} (identifying holomorphic functions with their associated germs around K), and

$$\mathfrak{g} = \varinjlim \mathfrak{g}_n \tag{63}$$

as a locally convex space (by definition). The direct limit (63) is compactly regular [16]. The Lie group G has an exponential map, given by

$$\exp_G: \mathfrak{g} \rightarrow G, \quad [\gamma] \mapsto [\exp_H \circ \gamma].$$

By the construction of the Lie group structure of G in [16], \exp_G is a local diffeomorphism at 0. Moreover, there is an open 0-neighbourhood $V \subseteq \mathfrak{g}$ such that the Baker-Campbell-Hausdorff series converges on $V \times V$ and makes it a complex analytic local Lie group, with $\exp_G(V)$ open and $\exp_G|_V$ both a diffeomorphism and a homomorphism of local Lie groups (see [16]). Then $V \cap \mathfrak{g}_n$ and hence also $\exp_G(V \cap \mathfrak{g}_n)$ is a local Lie group with Lie algebra \mathfrak{g}_n . As a consequence, the subgroup

$$G_n := \langle \exp_G(V \cap \mathfrak{g}_n) \rangle = \langle \exp_G(\mathfrak{g}_n) \rangle$$

is a Banach-Lie group with $L(G_n) = \mathfrak{g}_n$ and $\exp_{G_n} = \exp_G|_{\mathfrak{g}_n}$. Let $i_{n,m}: G_m \rightarrow G_n$, $j_{n,m}: \mathfrak{g}_m \rightarrow \mathfrak{g}_n$, $i_m: G_n \rightarrow G$ and $j_m: \mathfrak{g}_n \rightarrow \mathfrak{g}$ be the respective inclusion maps (for $n \geq m$). Since j_n and $j_{n,m}$ are continuous linear and hence complex analytic, \exp_{G_m} is a local diffeomorphism at 0 and \exp_G as well as \exp_{G_n} are complex analytic, we deduce from $i_m \circ \exp_{G_m} = \exp_G \circ j_m$ and $i_{n,m} \circ \exp_{G_m} = \exp_{G_n} \circ j_{n,m}$ that i_n and $i_{n,m}$ are complex analytic homomorphisms. Hence G is L_{rc}^∞ -regular, by Corollary 8.12.

The case $\mathbb{K} = \mathbb{R}$: Let \widetilde{M} be a complexification for M admitting an antiholomorphic involution $\tau: \widetilde{M} \rightarrow \widetilde{M}$ such that $M \subseteq \widetilde{M}$ and M is the fixed point set of τ (see [16]). If H is finite-dimensional, then H has a complexification $H_{\mathbb{C}}$ with $H \subseteq H_{\mathbb{C}}$ (see [10]). Then $\text{Germ}_{\mathbb{C}}(K, \widetilde{M}, H_{\mathbb{C}})$ is a complexification of $\text{Germ}_{\mathbb{R}}(K, M, H)$ (cf. [16]). Since $\text{Germ}_{\mathbb{C}}(K, \widetilde{M}, H_{\mathbb{C}})$ is L_{rc}^∞ -regular (by the complex case already treated), we deduce with Lemma 5.37 that $\text{Germ}_{\mathbb{R}}(K, M, H)$ is L_{rc}^∞ -regular with real analytic evolution Evol . It remains to show that $G := \text{Germ}_{\mathbb{R}}(K, M, H)$ is L_{rc}^∞ -regular when H is an arbitrary real Banach Lie group. To achieve this, let $U_1 \supseteq U_2 \supseteq \dots$ be a basis of open neighbourhoods of K in \widetilde{M} such that $U_n = \tau(U_n)$ for each n and each connected component of U_n meets M . Let $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \oplus i\mathfrak{h}$ be a complexification of $\mathfrak{h} := L(H)$ and $\sigma: \mathfrak{h}_{\mathbb{C}} \rightarrow \mathfrak{h}_{\mathbb{C}}$, $(x + iy) \mapsto x - iy$ for $x, y \in \mathfrak{h}$ be complex conjugation on $\mathfrak{h}_{\mathbb{C}}$. Then $\mathfrak{k}_n := \text{Hol}_b(U_n, \mathfrak{h}_{\mathbb{C}})$ is a complex Banach-Lie algebra and

$$\mathfrak{g}_n := \{\gamma \in \text{Hol}_b(U_n, \mathfrak{h}_{\mathbb{C}}) : \gamma = \sigma \circ \gamma \circ \tau\}$$

is a closed real Lie subalgebra of \mathfrak{k}_n such that $\mathfrak{k}_n = (\mathfrak{g}_n)_{\mathbb{C}}$ (cf. [16]). Identifying $\gamma \in \mathfrak{g}_n$ with the germ of $\gamma|_{U_n \cap M}: U_n \cap M \rightarrow \mathfrak{h}$, we can consider \mathfrak{g}_n as a Lie subalgebra of $\mathfrak{g} := \text{Germ}_{\mathbb{R}}(K, M, \mathfrak{h})$. Moreover,

$$\mathfrak{g} = \varinjlim \mathfrak{g}_n$$

(cf. [16]). We can now complete the proof as in the case $\mathbb{K} = \mathbb{C}$, replacing \mathbb{C} with \mathbb{R} there. \square

Taking $\mathbb{K} = \mathbb{R}$ and $M = K$, we get the following result as a special case, which subsumes Theorem F:

Corollary 8.20 *If M is a real analytic compact manifold and H a Banach-Lie group, then the Lie group $G := C^\omega(M, H)$ of all H -valued real analytic mappings on M is L_{rc}^∞ -regular. If H is a finite-dimensional Lie group, then $\text{Evol}_G: L_{rc}^\infty([0, 1], L(G)) \rightarrow AC_{L_{rc}^\infty}([0, 1], G)$ is real analytic.* \square

Remark 8.21 Real analyticity of Evol_G in Corollaries 8.19 and 8.20 also holds for all Banach-Lie groups G for which a complex Banach-Lie group $G_{\mathbb{C}}$ exists such that the inclusion map $G \subseteq G_{\mathbb{C}}$ is a real analytic homomorphism which makes $G_{\mathbb{C}}$ a complexification of G . Only this property is used in the proof.

Corollary 8.22 *The Lie group $C^\omega(\mathbb{R}, H)$ of all real analytic H -valued maps on \mathbb{R} is L_{rc}^∞ -regular, for each Banach-Lie group H .*

Proof. We recall from [16] that $C^\omega(\mathbb{R}, H)_* := \{\gamma \in C^\omega(\mathbb{R}, H) : \gamma(0) = e\}$ is a Lie subgroup of $C^\omega(\mathbb{R}, H)$ and

$$C^\omega(\mathbb{R}, H) = C^\omega(\mathbb{R}, H)_* \rtimes H$$

as a Lie group. Since H is L_{rc}^∞ -regular by Theorems C and A, we need only prove L_{rc}^∞ -regularity for $C^\omega(\mathbb{R}, H)$; then also $C^\omega(\mathbb{R}, H)$ will be L_{rc}^∞ -regular, by Theorem G. Since $\text{Germ}([-n, n], \mathbb{R}, H)$ is L_{rc}^∞ -regular by Corollary 8.19, also its Lie subgroup

$$\text{Germ}([-n, n], \mathbb{R}, H)_* := \{[\gamma] \in \text{Germ}([-n, n], \mathbb{R}, H) : \gamma(0) = e\}$$

is L_{rc}^∞ -regular, by Proposition 5.27 (as it is the kernel of the point evaluation $\text{Germ}([-n, n], \mathbb{R}, H) \rightarrow H$, $[\gamma] \mapsto \gamma(e)$, which is a smooth homomorphism to the L_{rc}^∞ -regular Lie group H). Now

$$C^\omega(\mathbb{R}, H)_* = \varprojlim \text{Germ}([-n, n], \mathbb{R}, H)_*$$

and

$$C^\omega(\mathbb{R}, H)_* \rightarrow C^\omega(\mathbb{R}, L(H)), \quad \gamma \mapsto \delta^\ell(\gamma)$$

is a projective limit chart (see [16]). Thus Proposition 7.14 shows that $C^\omega(\mathbb{R}, H)_*$ (and hence also $C^\omega(\mathbb{R}, H)$) is L_{rc}^∞ -regular. \square

9 Regularity properties of $\text{Diff}_c(\mathbb{R}^n)$, $\text{Diff}_K(\mathbb{R}^n)$ and $\text{Diff}(\mathbb{S}_1)$

After a brief introduction to the diffeomorphism groups $\text{Diff}_c(\mathbb{R}^n)$ and $\text{Diff}_K(\mathbb{R}^n)$, we prove the L^1 -regularity of $\text{Diff}_K(\mathbb{R}^n)$, $\text{Diff}(\mathbb{S}_1)$ and $\text{Diff}_c(\mathbb{R}^n)$.

9.1 If $U \subseteq \mathbb{R}^n$ is open and E a locally convex space, we endow $C(U, E)$ with the topology of uniform convergence on compact sets, determined by the seminorms

$$\|\cdot\|_{L,q}: C(U, E) \rightarrow [0, \infty[, \quad \gamma \mapsto \sup_{x \in L} q(\gamma(x))$$

for q in the set of continuous seminorms on E and L ranging through the compact subsets of U . If E and U are as before and $r \in \mathbb{N}_0 \cup \{\infty\}$, we endow the space $C^r(U, E)$ of all C^r -functions $\gamma: U \rightarrow E$ with the compact-open C^r -topology, i.e., the initial topology with respect to the maps

$$C^r(U, E) \rightarrow C(U, E), \quad \gamma \mapsto \frac{\partial^\alpha \gamma}{\partial x^\alpha}$$

for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq r$. Given a compact subset $K \subseteq U$, we give

$$C_K^r(U, E) := \{\gamma \in C^r(U, E) : \gamma|_{U \setminus K} = 0\}$$

the induced topology. It is the locally convex vector topology given by the seminorms

$$\|\gamma\|_{C^k,q} := \max_{|\alpha| \leq k} \|\partial^\alpha \gamma\|_{\mathcal{L}^\infty,q},$$

for all $k \in \mathbb{N}_0$ such that $k \leq r$ and all continuous seminorms q on E (with multi-indices $\alpha \in \mathbb{N}_0^n$). As usual, $C^r(U, E) = \bigcup_K C_K^r(U, E)$ is the locally convex direct limit of the spaces $C_K^r(U, E)$. If E is a Fréchet space, then also $C_K^r(U, E)$ is a Fréchet space. If E is a separable Fréchet space, then also $C_K^r(U, E)$ is separable.³⁰

³⁰Since $C_K^r(U, E)$ is isomorphic to $C_K^r(\mathbb{R}^n, E)$, it suffices to show that the Fréchet space $C^r(\mathbb{R}^n, E)$ is separable. Let $J := \{\alpha \in \mathbb{N}_0^n : |\alpha| \leq r\}$. We claim that $C(\mathbb{R}^n, E)$ is separable. If this is true, then the Fréchet space $C(\mathbb{R}^n, E)^J$ (with the product topology) will be separable. Since $C^r(\mathbb{R}^n, E) \rightarrow C(\mathbb{R}^n, E)^J$, $\gamma \mapsto (\partial^\alpha \gamma)_{|\alpha| \leq r}$ is a topological embedding, the separability of $C^r(\mathbb{R}^n, E)$ follows. To prove the claim, for $m \in \mathbb{N}$ let $(h_{m,k})_{k \in \mathbb{N}}$ be a partition of unity on \mathbb{R}^n subordinate to $(B_{1/m}(x))_{x \in \mathbb{R}^n}$ [66, I.8.6, Satz 3] (using balls with respect to some norm on \mathbb{R}^n). Let $D \subseteq E$ be a countable dense subset. Then the countable set $\{ah_{m,k} : m, k \in \mathbb{N}, a \in D\}$ is easily seen to be total in $C(\mathbb{R}^n, E)$ and thus $C(\mathbb{R}^n, E)$ is separable.

9.2 Let $C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be the space of all compactly supported, \mathbb{R}^n -valued smooth functions on \mathbb{R}^n and $\text{Diff}_c(\mathbb{R}^n)$ be the set of all diffeomorphisms $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ which are compactly supported in the sense that

$$\phi - \text{id}_{\mathbb{R}^n} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n).$$

It is known that $\text{Diff}_c(\mathbb{R}^n)$ is a Lie group under composition of diffeomorphisms, with neutral element $\text{id}_{\mathbb{R}^n}$ (cf. [51], see [26]). The set

$$\Omega := \{\phi - \text{id}_{\mathbb{R}^n} : \phi \in \text{Diff}_c(\mathbb{R}^n)\}$$

is open in $\text{Diff}_c(\mathbb{R}^n)$ and the map

$$\Phi: \text{Diff}_c(\mathbb{R}^n) \rightarrow \Omega, \quad \phi \mapsto \phi - \text{id}_{\mathbb{R}^n}$$

is a global chart; moreover,

$$\{\gamma \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) : \|\gamma'\|_{\mathcal{L}^\infty, \|\cdot\|_{op}} < 1\}$$

is an open subset of Ω (see [26]). Here $\|\alpha\|_{op}$ denotes the operator norm of a linear map $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with respect to the maximum norm $\|\cdot\|_\infty$ on \mathbb{R}^n . We can make Ω a Lie group in such a way that Φ becomes an isomorphism of Lie groups: Its group multiplication is given by

$$\gamma * \eta := \Phi(\Phi^{-1}(\gamma) \circ \Phi^{-1}(\eta)) = (\text{id}_{\mathbb{R}^n} + \gamma) \circ (\text{id}_{\mathbb{R}^n} + \eta) - \text{id}_{\mathbb{R}^n} = \eta + \gamma \circ (\text{id}_{\mathbb{R}^n} + \eta)$$

for $\gamma, \eta \in \Omega$, and the constant function 0 is the neutral element. Note that Φ takes $\text{Diff}_K(\mathbb{R}^n)$ onto $\Omega \cap C_K^\infty(\mathbb{R}^n, \mathbb{R}^n) =: \Omega_K$. Hence $\text{Diff}_K(\mathbb{R}^n)$ is a Lie subgroup of $\text{Diff}_c(\mathbb{R}^n)$ modelled on $C_K^\infty(\mathbb{R}^n, \mathbb{R}^n)$ which has

$$\Phi_K: \text{Diff}_K(\mathbb{R}^n) \rightarrow \Omega_K, \quad \phi \mapsto \phi - \text{id}_{\mathbb{R}^n}$$

as a global chart. Again, Ω_K can be made a Lie group isomorphic to $\text{Diff}_K(\mathbb{R}^n)$ using the multiplication $*$. To see that $\text{Diff}_c(\mathbb{R}^n)$ and $\text{Diff}_K(\mathbb{R}^n)$ are L^1 -regular, we need only show that Ω and Ω_K are L^1 -regular.

9.3 As usual for tangent bundles of open subsets of locally convex spaces, we have³¹

$$T\Omega = \Omega \times C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \quad \text{and} \quad T\Omega_K = \Omega_K \times C_K^\infty(\mathbb{R}^n, \mathbb{R}^n).$$

³¹Thus, we are using the addition of the locally convex space to trivialize the tangent bundle, *not* left or right multiplication in the Lie group $(\Omega, *)$.

For fixed $\eta \in \Omega$, right translation with η is the map

$$\rho_\eta: \Omega \rightarrow \Omega, \quad \gamma \mapsto \eta + \gamma \circ (\text{id}_{\mathbb{R}^n} + \eta)$$

which is the restriction of the affine linear map $C_c^\infty(\mathbb{R}, \mathbb{R}^n) \rightarrow C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ given by the same formula, which is continuous (see [26]). Hence

$$d\rho_\eta(\gamma, \gamma_1) = \gamma_1 \circ (\text{id}_{\mathbb{R}^n} + \eta) \quad \text{for all } (\gamma, \gamma_1) \in \Omega \times C_c^\infty(\mathbb{R}^n, \mathbb{R}^n).$$

We identify the Lie algebra $L(\Omega) = T_0\Omega = \{0\} \times C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Let us calculate the product of $(\gamma, \gamma_1) \in T\Omega$ and $\eta \in \Omega$ (identified with $0_\eta \in T_\eta\Omega$) in the tangent group $T\Omega$. We have

$$(\gamma, \gamma_1) \cdot \eta = T\rho_\eta(\gamma, \gamma_1) = (\gamma * \eta, \gamma_1 \circ (\text{id}_{\mathbb{R}^n} + \eta)).$$

Likewise for Ω_K .

9.4 Given $\gamma \in \mathcal{L}^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$, we want to find a continuous function $\eta: [0, 1] \rightarrow \Omega$ which is a Carathéodory solution to

$$(\eta(t), \eta'(t)) = (0, \gamma(t)) \cdot \eta(t) = (\eta(t), \gamma(t) \circ (\text{id}_{\mathbb{R}^n} + \eta(t))) \quad (t \in [0, 1])$$

in $T\Omega = \Omega \times C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\eta(0) = 0$. As a differential equation in a locally convex space, this requires

$$\eta'(t) = \gamma(t) \circ (\text{id}_{\mathbb{R}^n} + \eta(t))$$

and hence that

$$\eta(t) = \int_0^t \gamma(s) \circ (\text{id}_{\mathbb{R}^n} + \eta(s)) ds \quad \text{for all } t \in [0, 1] \quad (64)$$

in $C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. We shall see that, as a function of s , the integrand in (64) is an element of $\mathcal{L}^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$ for each $\eta \in C([0, 1], \Omega)$ (Lemma 9.7); thus validity of (64) implies that $\eta \in AC_{L^1}([0, 1], \Omega)$ and $\eta = \text{Evol}^r([\gamma])$.

9.5 Likewise, if $\gamma \in \mathcal{L}^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^n))$, we wish to find a continuous map $\eta: [0, 1] \rightarrow \Omega_K$ such that (64) holds. Then $\eta \in AC_{L^1}([0, 1], \Omega_K)$ (see Lemma 9.7) and $\eta = \text{Evol}^r([\gamma])$.

In 9.5, we need to make sure that the integrand of (64) is an element of $\mathcal{L}^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^n))$ as a function of s . Moreover, in both 9.4 and 9.5, the smooth dependence of $\text{Evol}^r([\gamma])$ on $[\gamma]$ remains to be shown. To enable these tasks, we now provide several preparatory lemmas devoted to measurability and differentiability properties in related situations. The point evaluation $\text{ev}_x: C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R}^n$, $f \mapsto f(x)$ is continuous linear for each $x \in \mathbb{R}^n$, and these point evaluations separate points on $C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Hence, if the integrand in (64) is an element of $\mathcal{L}^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$, then (64) holds if and only if the continuous functions $\eta_x := \text{ev}_x \circ \eta: [0, 1] \rightarrow \mathbb{R}^n$ satisfy

$$\eta_x(t) = \int_0^t \gamma(s)(x + \eta_x(s)) ds \quad \text{for all } t \in [0, 1],$$

for all $x \in \mathbb{R}^n$. Setting $\zeta_x(t) := x + \eta_x(t)$, the latter is equivalent to

$$\zeta_x(t) = x + \int_0^t \gamma(s)(\zeta_x(s)) ds \quad \text{for all } t \in [0, 1], \quad (65)$$

meaning that $\zeta_x: [0, 1] \rightarrow \mathbb{R}^n$ is a Carathéodory solution to

$$\zeta'_x(t) = \gamma(t)(\zeta_x(t)), \quad \zeta_x(0) = x.$$

Our strategy is to discuss the solutions to (65), and their dependence on (γ, x) .

9.6 By the preceding, if $\eta: [0, 1] \rightarrow \Omega$ is continuous and the integrand of (64) is an element of $\mathcal{L}^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$, then the validity of (65) for all $x \in \mathbb{R}^n$ implies the validity of (64). Likewise, if $\eta: [0, 1] \rightarrow \Omega_K$ is continuous and the integrand of (64) is in $\mathcal{L}^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^n))$, then the validity of (65) for all $x \in \mathbb{R}^n$ implies the validity of (64).

Lemma 9.7 *If $\gamma \in \mathcal{L}^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$ and $\eta \in C([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$, then*

$$(s \mapsto \gamma(s) \circ (\text{id}_{\mathbb{R}^n} + \eta(s))) \in \mathcal{L}^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)).$$

If $\gamma \in \mathcal{L}^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^n))$ and $\eta \in C([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^n))$, then

$$(s \mapsto \gamma(s) \circ (\text{id}_{\mathbb{R}^n} + \eta(s))) \in \mathcal{L}^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^n)).$$

Proof. The map

$$f: C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \rightarrow C_c^\infty(\mathbb{R}^n, \mathbb{R}^n), \quad f(\sigma, \tau) := \tau \circ (\text{id}_{\mathbb{R}^n} + \sigma)$$

is smooth (see [26]), and $f(\sigma, \bullet)$ is linear for each $\sigma \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Hence, by Lemma 2.1 (b),

$$f \circ (\eta, \gamma) \in \mathcal{L}^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$$

for all $\gamma \in C([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$ and $\eta \in \mathcal{L}^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$, where

$$(f \circ (\eta, \gamma))(s) = f(\eta(s), \gamma(s)) = \gamma(s) \circ (\text{id}_{\mathbb{R}^n} + \eta(s)).$$

Given a compact set $K \subseteq \mathbb{R}^n$, the map f restricts to a smooth map

$$f_K: C_K^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C_K^\infty(\mathbb{R}^n, \mathbb{R}^n) \rightarrow C_K^\infty(\mathbb{R}^n, \mathbb{R}^n), \quad (\sigma, \tau) \mapsto \tau \circ (\text{id}_{\mathbb{R}^n} + \sigma),$$

and again Lemma 2.1 (b) can be applied. \square

L^1 -regularity of $\text{Diff}_K(\mathbb{R}^n)$

Lemma 9.8 *Let $K \subseteq \mathbb{R}^n$ be compact, $m \in \mathbb{N}_0$ and $\gamma \in \mathcal{L}^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^m))$. Then*

$$\widehat{\gamma}: [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad \widehat{\gamma}(t, x) := \gamma(t)(x)$$

is measurable. For any measurable function $\zeta: [0, 1] \rightarrow \mathbb{R}^n$, define a function

$$(\widehat{\gamma})_*(\zeta): [0, 1] \rightarrow \mathbb{R}^m$$

via $(\widehat{\gamma})_(\zeta)(t) := \widehat{\gamma}(t, \zeta(t)) = \gamma(t)(\zeta(t))$. Then $(\widehat{\gamma})_*(\zeta) \in \mathcal{L}^1([0, 1], \mathbb{R}^m)$.*

Proof. Since \mathbb{R}^n is second countable, the Borel σ -algebra $\mathcal{B}(C_K^\infty(\mathbb{R}^n, \mathbb{R}^m) \times \mathbb{R}^n)$ coincides with the product σ -algebra $\mathcal{B}(C_K^\infty(\mathbb{R}^n, \mathbb{R}^m)) \otimes \mathcal{B}(\mathbb{R}^n)$ (see 1.6 (f)). Therefore the map

$$\gamma \times \text{id}_{\mathbb{R}^n}: [0, 1] \times \mathbb{R}^n \rightarrow C_K^\infty(\mathbb{R}^n, \mathbb{R}^m) \times \mathbb{R}^n, \quad (t, x) \mapsto (\gamma(t), x)$$

is Borel measurable. The evaluation map

$$\varepsilon: C_K^\infty(\mathbb{R}^n, \mathbb{R}^m) \times \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad \varepsilon(f, x) := f(x)$$

is C^∞ (see, e.g., [25] or [38]), hence continuous and hence measurable. Thus $\widehat{\gamma} = \varepsilon \circ (\gamma \times \text{id}_{\mathbb{R}^n})$ is measurable. As $(\text{id}_{[0, 1]}, \zeta): [0, 1] \rightarrow [0, 1] \times \mathbb{R}^n, t \mapsto (t, \zeta(t))$

is measurable, also the composition $(\hat{\gamma})_*(\zeta) = \hat{\gamma} \circ (\text{id}_{[0,1]}, \zeta)$ is measurable. On \mathbb{R}^m , we use the maximum-norm $\|\cdot\|_\infty$, giving rise to a continuous norm $q := \|\cdot\|_{\mathcal{L}^\infty, \|\cdot\|_\infty}$ on $C_K^\infty(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathcal{L}^\infty(\mathbb{R}^n, \mathbb{R}^m)$, which in turn gives rise to a continuous seminorm $\|\cdot\|_{\mathcal{L}^1, q}$ on $\mathcal{L}^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^m))$. Now

$$\|(\hat{\gamma})_*(\zeta)(t)\|_\infty = \|\gamma(t)(\zeta(t))\|_\infty \leq \|\gamma(t)\|_{\mathcal{L}^\infty, \|\cdot\|_\infty} = q(\gamma(t)), \quad (66)$$

whence $\int_0^1 \|(\hat{\gamma})_*(\zeta)(t)\|_\infty dt \leq \int_0^1 q(\gamma(t)) dt = \|\gamma\|_{\mathcal{L}^1, q} < \infty$ and hence $(\hat{\gamma})_*(\zeta) \in \mathcal{L}^1([0, 1], \mathbb{R}^m)$, with $\|(\hat{\gamma})_*(\zeta)\|_{\mathcal{L}^1, \|\cdot\|_\infty} \leq \|\gamma\|_{\mathcal{L}^1, q}$. \square

With notation as in the preceding lemma, we have:

Lemma 9.9 *For each $m \in \mathbb{N}$ and each compact subset $K \subseteq \mathbb{R}^n$, the map*

$$\Phi_m: L^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^m)) \times C([0, 1], \mathbb{R}^n) \rightarrow L^1([0, 1], \mathbb{R}^m),$$

$\Phi_m([\gamma], \zeta) := [(\hat{\gamma})_*(\zeta)]$, *is smooth.*

Proof. Since $\Phi_m(\gamma, \zeta)$ is linear in γ , it suffices to show that Φ_m is $C^{0,\infty}$ (see 1.56 (b) and (a)). We show by induction that Φ_m is $C^{0,k}$ for each $k \in \mathbb{N}_0$. Let $k = 0$ first; we have to show that Φ_m is continuous. From the preceding proof, we know that

$$\|\Phi_m(\gamma, \zeta)\|_{L^1, \|\cdot\|_\infty} \leq \|\gamma\|_{L^1, q}$$

with $q := \|\cdot\|_{\mathcal{L}^\infty, \|\cdot\|_\infty}$. Consider the map $D: C_K^\infty(\mathbb{R}^n, \mathbb{R}^m) \rightarrow C_K^\infty(\mathbb{R}^n, \mathbb{R}^{m \times n})$ such that $D(f)(x) := f'(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{m \times n}$ is the Jacobi matrix of f at x . Then D is continuous linear, entailing that $p := \|\cdot\|_{\mathcal{L}^\infty, \|\cdot\|_{op}} \circ D$ is a continuous seminorm on $C_K^\infty(\mathbb{R}^n, \mathbb{R}^m)$. Thus $p(f) = \sup_{x \in \mathbb{R}^n} \|f'(x)\|_{op}$ for $f \in C_K^\infty(\mathbb{R}^n, \mathbb{R}^m)$. Let $\tilde{\gamma} \in \mathcal{L}^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^m))$ and $\gamma := [\tilde{\gamma}] \in L^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^m))$. We have for all $\eta, \eta_1 \in C([0, 1], \mathbb{R}^n)$ and $t \in [0, 1]$

$$\tilde{\gamma}(t)(\eta(t)) - \tilde{\gamma}(t)(\eta_1(t)) = \int_0^1 (D(\tilde{\gamma}(t))(\eta_1(t) + s(\eta(t) - \eta_1(t))).(\eta(t) - \eta_1(t))) ds \quad (67)$$

and thus

$$\begin{aligned} & \|\tilde{\gamma}(t)(\eta(t)) - \tilde{\gamma}(t)(\eta_1(t))\|_\infty \\ & \leq \int_0^1 \|D(\tilde{\gamma}(t))(\eta_1(t) + s(\eta(t) - \eta_1(t))).(\eta(t) - \eta_1(t))\|_\infty ds \\ & \leq \|D(\tilde{\gamma}(t))\|_{\mathcal{L}^\infty, \|\cdot\|_{op}} \|\eta - \eta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty} \\ & = p(\tilde{\gamma}(t)) \|\eta - \eta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty}. \end{aligned} \quad (68)$$

Integrating over t , we deduce that

$$\|\Phi_m(\gamma)(\eta) - \Phi_m(\gamma)(\eta_1)\|_{L^1, \|\cdot\|_\infty} \leq \|\gamma\|_{L^1, p} \|\eta - \eta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty}.$$

As a consequence,

$$\begin{aligned} & \|\Phi_m(\gamma, \zeta) - \Phi_m(\gamma_1, \zeta_1)\|_{L^1, \|\cdot\|_\infty} \\ & \leq \|\Phi_m(\gamma, \zeta) - \Phi_m(\gamma, \zeta_1)\|_{L^1, \|\cdot\|_\infty} + \|\Phi_m(\gamma, \zeta_1) - \Phi_m(\gamma_1, \zeta_1)\|_{L^1, \|\cdot\|_\infty} \\ & \leq \|\gamma\|_{L^1, p} \|\eta - \eta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty} + \|\gamma - \gamma_1\|_{L^1, q} \end{aligned}$$

for all $\gamma_1 \in L^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^m))$ and γ, η, η_1 as before, which can be made arbitrarily small for γ_1 close to γ and η_1 close to η . Thus Φ_m is continuous at each (γ, η) and thus Φ_m is continuous.

Let $k \in \mathbb{N}$ now and assume that Φ_m is $C^{0, k-1}$ for each $m \in \mathbb{N}$. Let $\gamma = [\tilde{\gamma}] \in L^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^m))$ with $\tilde{\gamma} \in \mathcal{L}^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^m))$ and $\eta, \eta_1 \in C([0, 1], \mathbb{R}^n)$. To calculate $d_2\Phi_m(\gamma, \eta; \eta_1)$, we consider the corresponding directional difference quotients first. For $t \in [0, 1]$ and $\tau \in \mathbb{R} \setminus \{0\}$, we have

$$\frac{\tilde{\gamma}(t)(\eta(t) + \tau\eta_1(t)) - \tilde{\gamma}(t)(\eta(t))}{\tau} = \int_0^1 (D(\tilde{\gamma}(t))(\eta(t) + s\tau\eta_1(t)) \cdot \eta_1(t)) ds \quad (69)$$

by (67). The map

$$\alpha: L^1([0, 1], \mathbb{R}^{m \times n}) \rightarrow L^1([0, 1], \mathbb{R}^m), \quad \alpha([f]) := [s \mapsto f(s)\eta_1(s)]$$

for $f \in \mathcal{L}^1([0, 1], \mathbb{R}^{m \times n})$, $s \in [0, 1]$ (given pointwise by multiplication of matrices and vectors) is linear and continuous, with $\|\alpha\|_{op} \leq \|\eta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty}$. We abbreviate $\bar{\gamma} := D \circ \tilde{\gamma}$ and identify $\mathbb{R}^{m \times n}$ with \mathbb{R}^{mn} . Then the mapping $h: \mathbb{R} \times [0, 1] \rightarrow L^1([0, 1], \mathbb{R}^m)$,

$$h(\tau, s) := \alpha([\widehat{\tilde{\gamma}}]_*(\eta + s\tau\eta_1)) = \alpha(\Phi_{mn}([\bar{\gamma}], \eta + s\tau\eta_1))$$

is continuous, by induction, and we record that

$$h(0, s) = \alpha(\Phi_{mn}([\bar{\gamma}], \eta)) = \Phi_{mn}([\bar{\gamma}], \eta)\eta_1$$

is independent of $s \in [0, 1]$. Now the theorem on parameter-dependent integrals (see 1.18) shows that

$$g: \mathbb{R} \rightarrow L^1([0, 1], \mathbb{R}^m), \quad g(\tau) := \int_0^1 h(\tau, s) ds$$

is continuous. We claim that

$$g(\tau) = \frac{\Phi_m(\gamma, \eta + \tau\eta_1) - \Phi_m(\gamma, \eta)}{\tau} \quad \text{for all } \tau \in \mathbb{R} \setminus \{0\}.$$

If this is true, then the continuity of g implies that the limit as $\tau \rightarrow 0$ exists; we have

$$\begin{aligned} d_2\Phi_m(\gamma, \eta; \eta_1) &= \lim_{\tau \rightarrow 0} \frac{\Phi_m(\gamma, \eta + \tau\eta_1) - \Phi_m(\gamma, \eta)}{\tau} \\ &= \lim_{\tau \rightarrow 0} g(\tau) = g(0) = \int_0^1 h(0, s) ds = \Phi_{mn}([\bar{\gamma}], \eta)\eta_1. \end{aligned}$$

The map

$$\beta: L^1([0, 1], \mathbb{R}^{m \times n}) \times C([0, 1], \mathbb{R}^n) \rightarrow L^1([0, 1], \mathbb{R}^m), \quad \beta([f], g) := [t \mapsto f(t)g(t)]$$

given by pointwise multiplication of matrices and vectors is continuous bilinear with $\|\beta\|_{\text{op}} \leq 1$, and hence smooth. By the preceding, we have

$$d_2\Phi_m(\gamma, \eta; \eta_1) = \beta(\Phi_{mn}([\bar{\gamma}], \eta), \eta_1). \quad (70)$$

The map

$$L^1([0, 1], D): L^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^m)) \rightarrow L^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^{m \times n}))$$

sending $\gamma = [\tilde{\gamma}]$ to $[\bar{\gamma}] = [D \circ \tilde{\gamma}]$ is continuous linear. The map Φ_{mn} is $C^{0,k-1}$ by induction. Hence also $(\gamma, \eta) \mapsto \Phi_{mn}([\bar{\gamma}], \eta)$ is $C^{0,k-1}$ (see 1.56 (c)). Looking at the right hand side of (70), we deduce with 1.56 (d) that $d_2\Phi_m$ is $C^{0,k-1,\infty}$ as a function of (γ, η, η_1) and hence (by 1.56 (e)) $C^{0,k-1}$ as a function of $(\gamma, (\eta, \eta_1))$. Hence Φ_m is $C^{0,k}$, by 1.56 (f).

To prove the claim made above, we consider the continuous linear functionals

$$I_{\lambda, \theta}: L^1([0, 1], \mathbb{R}^m) \rightarrow \mathbb{R}, \quad I_{\lambda, \theta}([f]) := \int_0^1 \lambda(f(t))\theta(t) dt$$

for $\theta \in \mathcal{L}^\infty([0, 1], \mathbb{R})$ and λ in the dual space $(\mathbb{R}^m)'$. Then

$$\begin{aligned}
I_{\lambda, \theta}(g(\tau)) &= \int_0^1 I_{\lambda, \theta}(h(\tau, s)) ds \\
&= \int_0^1 \int_0^1 \lambda(D(\tilde{\gamma}(t))(\eta(t) + s\tau\eta_1(t))\eta_1(t))\theta(t) dt ds \\
&= \int_0^1 \lambda\left(\int_0^1 D(\tilde{\gamma}(t))(\eta(t) + s\tau\eta_1(t))\eta_1(t) ds\right) \theta(t) dt \\
&= \int_0^1 \lambda\left(\frac{\tilde{\gamma}(t)(\eta(t) + \tau\eta_1(t)) - \tilde{\gamma}(t)(\eta(t))}{\tau}\right) \theta(t) dt \\
&= I_{\lambda, \theta}\left(\frac{\Phi_n(\gamma, \eta + \tau\eta_1) - \Phi_n(\gamma, \eta)}{\tau}\right),
\end{aligned}$$

using (69) for the penultimate equality and Fubini's Theorem for the third equality (justified by Lemma 9.10). As the $I_{\lambda, \theta}$ separate points on $L^1([0, 1], \mathbb{R}^m)$, the claim is established. \square

We hasten to check that the hypotheses of Fubini's Theorem were satisfied in the preceding situation.

Lemma 9.10 *The function $f: [0, 1]^2 \rightarrow \mathbb{R}$,*

$$(t, s) \mapsto \lambda(D(\tilde{\gamma}(t))(\eta(t) + s\tau\eta_1(t))\eta_1(t))\theta(t)$$

is in $\mathcal{L}^1([0, 1]^2, \mathbb{R})$ with respect to Lebesgue-Borel measure on $[0, 1]^2$.

Proof. To see that f is measurable, write $\tilde{\gamma}(t) = (\tilde{\gamma}_1(t), \dots, \tilde{\gamma}_m(t))$ (identifying $C_K^\infty(\mathbb{R}^n, \mathbb{R}^m)$ with $C_K^\infty(\mathbb{R}^n, \mathbb{R}^m)$). Then $t \mapsto \frac{\partial \tilde{\gamma}_i(t)}{\partial x_j}$ is an element of $\mathcal{L}^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}))$ for each $j \in \{1, \dots, n\}$. Write $\eta_1 = (\eta_{1,1}, \dots, \eta_{1,n})$ with continuous functions $\eta_{1,j}: [0, 1] \rightarrow \mathbb{R}$ for $j \in \{1, \dots, n\}$. There are $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that $\lambda(x_1, \dots, x_m) = \lambda_1 x_1 + \dots + \lambda_m x_m$. The evaluation map

$$\varepsilon: C([0, 1], \mathbb{R}^n) \times [0, 1] \rightarrow \mathbb{R}^n, \quad (\kappa, t) \mapsto \kappa(t)$$

is continuous and hence measurable. The map $[0, 1]^2 \rightarrow C([0, 1], \mathbb{R}^n) \times [0, 1]$, $(s, t) \mapsto (\eta + s\tau\eta_1, t)$ is continuous and hence measurable. Now the formula

$$f(t, s) = \sum_{i=1}^m \sum_{j=1}^n \lambda_i \frac{\partial \tilde{\gamma}_i(t)}{\partial x_j} (\varepsilon(\eta + s\tau\eta_1, t)) \eta_{1,j}(t) \theta(t)$$

shows that f is measurable, being a sum of products of measurable real-valued functions. Using Fubini's theorem for non-negative measurable functions on $[0, 1]^2$, we find that

$$\begin{aligned}
& \int_{[0,1]^2} |f(t, s)| d\lambda_2(t, s) \\
&= \int_0^1 \int_0^1 |f(t, s)| ds dt \\
&\leq \|\theta\|_{\mathcal{L}^\infty} \sum_{i=1}^m \sum_{j=1}^n \|\eta_{1,j}\|_{\mathcal{L}^\infty} |\lambda_i| \int_0^1 \int_0^1 \underbrace{\left| \frac{\partial \tilde{\gamma}_i(t)}{\partial x_j} (\varepsilon(\eta + s\tau\eta_1, t)) \right|}_{\leq \|\frac{\partial \tilde{\gamma}_i(t)}{\partial x_j}\|_{\mathcal{L}^\infty}} ds dt \\
&\leq \int_0^1 \left\| \frac{\partial \tilde{\gamma}_i}{\partial x_j} \right\|_{\mathcal{L}^\infty} dt = \|\partial/\partial x_j \circ \tilde{\gamma}_i\|_{\mathcal{L}^1, p} < \infty
\end{aligned}$$

with $p := \|\cdot\|_{\mathcal{L}^\infty} := \|\cdot\|_{\mathcal{L}^\infty, |\cdot|}$. □

9.11 Consider $D: C_K^\infty(\mathbb{R}^n, \mathbb{R}^n) \rightarrow C_K^\infty(\mathbb{R}^n, \mathbb{R}^{n \times n})$, $f \mapsto f'$ and the continuous seminorm $p := \|\cdot\|_{\mathcal{L}^\infty, \|\cdot\|_{op}} \circ D$ on $C_K^\infty(\mathbb{R}^n, \mathbb{R}^n)$; thus $p(f) = \sup_{x \in \mathbb{R}^n} \|f'(x)\|_{op}$ for $f \in C_K^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Fix $L \in]0, 1[$. Then

$$Q_K := \{[\gamma] \in L^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^n)) : \|\gamma\|_{L^1, p} < L\}$$

is an open 0-neighbourhood in $L^1([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^n))$.

We define a map $\Psi_K: Q_K \times \mathbb{R}^n \times C([0, 1], \mathbb{R}^n) \rightarrow C([0, 1], \mathbb{R}^n)$ via

$$\Psi_K([\gamma], x, \kappa)(t) := x + \int_0^t \gamma(s)(\kappa(s)) ds$$

for $[\gamma] \in Q_K$ with $\gamma \in \mathcal{L}^1([0, 1], C_K^\infty(\mathbb{R}, \mathbb{R}^n))$, $x \in \mathbb{R}^n$, $\kappa \in C([0, 1], \mathbb{R}^n)$ and $t \in [0, 1]$.

Lemma 9.12 *The map $\Psi_K: Q_K \times \mathbb{R}^n \times C([0, 1], \mathbb{R}^n) \rightarrow C([0, 1], \mathbb{R}^n)$ is smooth and defines a uniform family of contractions in the final variable, in the sense that*

$$\text{Lip}(\Psi_K([\gamma], x, \bullet)) \leq L$$

for all $[\gamma] \in Q_K$ and $x \in \mathbb{R}^n$.

Proof. For $x \in \mathbb{R}^n$ let $c_x: [0, 1] \rightarrow \mathbb{R}^n$ be the constant function $t \mapsto x$. The map $\mathbb{R}^n \rightarrow C([0, 1], \mathbb{R}^n)$, $x \mapsto c_x$ is continuous linear and hence smooth. Moreover, the operator

$$J: L^1([0, 1], \mathbb{R}^n) \rightarrow C([0, 1], \mathbb{R}^n)$$

determined by $J([f])(t) := \int_0^t f(s) ds$ is continuous and linear and hence smooth. Now the formula

$$\Psi_K([\gamma], x, \kappa) = c_x + J(\Phi_n([\gamma], \kappa))$$

(with the smooth map Φ_n from Lemma 9.9) shows that Ψ_K is smooth. Given $\eta, \eta_1 \in C([0, 1], \mathbb{R}^n)$, we deduce from (68) that

$$\begin{aligned} \|\Psi_K([\gamma], x, \eta) - \Psi_K([\gamma], x, \eta_1)\|_\infty &= \sup_{t \in [0, 1]} \left\| \int_0^t \gamma(s)(\eta(s)) - \gamma(s)(\eta_1(s)) ds \right\|_\infty \\ &\leq \sup_{t \in [0, 1]} \int_0^t \underbrace{\|\gamma(s)(\eta(s)) - \gamma(s)(\eta_1(s))\|_\infty}_{\leq p(\gamma(t))\|\eta - \eta_1\|_{L^\infty}} ds \\ &\leq \|\gamma\|_{\mathcal{L}^1, p} \|\eta - \eta_1\|_{L^\infty} \leq L \|\eta - \eta_1\|_{L^\infty} \end{aligned}$$

with p as in 9.11. This ends the proof. \square

For each $([\gamma], x) \in Q_K \times \mathbb{R}^n$, the contraction

$$\Psi_K([\gamma], x, \bullet): C([0, 1], \mathbb{R}^n) \rightarrow C([0, 1], \mathbb{R}^n)$$

of the Banach space $C([0, 1], \mathbb{R}^n)$ has a unique fixed point $\zeta_{[\gamma], x} \in C([0, 1], \mathbb{R}^n)$, by Banach's Contraction Principle; thus

$$\Psi_K([\gamma], x, \zeta_{[\gamma], x}) = \zeta_{[\gamma], x}. \quad (71)$$

Since Ψ_K is smooth, Lemma 6.2 shows that also the map

$$Q_K \times \mathbb{R}^n \rightarrow C([0, 1], \mathbb{R}^n), \quad ([\gamma], x) \mapsto \zeta_{[\gamma], x}$$

is smooth. Define $F_K([\gamma])(t)(x) := \zeta_{[\gamma], x}(t)$ for $[\gamma] \in Q_K$, $x \in \mathbb{R}^n$ and $t \in [0, 1]$. Using the exponential laws from [2], we deduce:

$$(a) \quad F_K([\gamma])(t) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \text{ for all } [\gamma] \in Q_K \text{ and } t \in [0, 1];$$

- (b) $F_K([\gamma]) \in C([0, 1], C^\infty(\mathbb{R}^n, \mathbb{R}^n))$ for all $[\gamma] \in Q_K$;
- (c) $F_K: Q_K \rightarrow C([0, 1], C^\infty(\mathbb{R}^n, \mathbb{R}^n))$ is smooth.

As a consequence, also the map

$$E_K: Q_K \rightarrow C([0, 1], C^\infty(\mathbb{R}^n, \mathbb{R}^n)), \quad [\gamma] \mapsto F_K([\gamma]) - I$$

is smooth, where $I: [0, 1] \rightarrow C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is the constant function $t \mapsto \text{id}_{\mathbb{R}^n}$. If $x \in \mathbb{R}^n \setminus K$, then $\zeta_{[\gamma], x}$ is the fixed point of the map $\Psi_K([\gamma], x, \bullet)$ determined by

$$\Psi_K([\gamma], x, \kappa)(t) = x + \int_0^t \gamma(s)(\kappa(s)) ds.$$

Since $\gamma(s)(x) = 0$ for each $s \in [0, 1]$, also the constant map c_x is a fixed point and thus $\zeta_{[\gamma], x} = c_x$ by uniqueness of the latter. As a consequence,

$$E_K([\gamma])(t) \in C_K^\infty(\mathbb{R}^n, \mathbb{R}^n)$$

for each $t \in [0, 1]$ and thus E_K can be considered as a smooth map

$$E_K: Q_K \rightarrow C([0, 1], C_K^\infty(\mathbb{R}^n, \mathbb{R}^n)).$$

Since $E_K(0) = E_\emptyset(0) = 0$ and E_K is continuous, there is an open 0-neighbourhood $P_K \subseteq Q_K$ such that

$$E_K(P_K) \subseteq \Omega_K.$$

For each $[\gamma] \in P_K$, we have $\eta := E_K([\gamma]) \in C([0, 1], \Omega_K)$ and $\zeta_x(t) := x + \gamma(t)(x) = F_K([\gamma])(t)(x) = \zeta_{[\gamma], x}$ satisfies (65) by (71). Hence $\eta: [0, 1] \rightarrow \Omega_K$ satisfies (64) by the discussion in (9.4) and thus $\eta = \text{Evol}_{\Omega_K}([\gamma])$. We now deduce from Proposition 5.25 and Lemma 5.26 that $(\Omega_K, *)$ (and hence also $\text{Diff}_K(\mathbb{R}^n)$) is L^1 -regular.

L^1 -regularity of $\text{Diff}(\mathbb{S}_1)$

In the Fréchet space

$$C_{2\pi}^\infty(\mathbb{R}, \mathbb{R}) := \{\gamma \in C^\infty(\mathbb{R}, \mathbb{R}) : (\forall x \in \mathbb{R}) \gamma(x + 2\pi) = \gamma(x)\},$$

the set

$$\Omega_{2\pi} := \{\gamma \in C_{2\pi}^\infty(\mathbb{R}, \mathbb{R}) : (\forall x \in [0, 2\pi]) \gamma'(x) > -1\}$$

is open and convex (hence simply connected). It is a well-known fact the $\Omega_{2\pi}$ is the universal covering group of the identity component $\text{Diff}(\mathbb{S}_1)_0$ of $\text{Diff}(\mathbb{S}_1)$, with the group multiplication

$$\Omega_{2\pi} \times \Omega_{2\pi} \rightarrow \Omega_{2\pi}, \quad (\gamma, \eta) \mapsto \eta + \gamma \circ (\text{id}_{\mathbb{R}} + \eta)$$

(see, e.g., [38]). The universal covering map takes $\gamma \in \Omega_{2\pi}$ to $\phi_\gamma \in \text{Diff}(\mathbb{S}_1)_0$,

$$\phi_\gamma(e^{it}) := e^{i\gamma(t)} e^{it} = e^{i(t+\gamma(t))} \quad \text{for all } t \in \mathbb{R}.$$

Setting $n = 1$ and replacing Ω_K with $\Omega_{2\pi}$ and $C_K^\infty(\mathbb{R}^n, \mathbb{R}^m)$ with $C_{2\pi}^\infty(\mathbb{R}, \mathbb{R}^m)$ in the preceding discussion of $\text{Diff}_K(\mathbb{R}^n) \cong \Omega_K$, we see that $\Omega_{2\pi}$ (and hence $\text{Diff}(\mathbb{S}_1)$) is L^1 -regular.

L^1 -regularity of $\text{Diff}_c(\mathbb{R}^n)$

Lemma 9.13 *Let $U \subseteq \mathbb{R}^n$ be an open set and $V \subseteq U$ be an open, convex subset with compact closure $\overline{V} \subseteq U$. Let $m \in \mathbb{N}_0$ and $\gamma \in \mathcal{L}^1([0, 1], C^\infty(U, \mathbb{R}^m))$. Then the map*

$$\widehat{\gamma} : [0, 1] \times U \rightarrow \mathbb{R}^m, \quad \widehat{\gamma}(t, x) := \gamma(t)(x)$$

is measurable. For any measurable function $\zeta : [0, 1] \rightarrow V$, define a function

$$(\widehat{\gamma})_*(\zeta) : [0, 1] \rightarrow \mathbb{R}^m$$

via $(\widehat{\gamma})_(\zeta)(t) := \widehat{\gamma}(t, \zeta(t)) = \gamma(t)(\zeta(t))$. Then $(\widehat{\gamma})_*(\zeta) \in \mathcal{L}^1([0, 1], \mathbb{R}^m)$.*

Proof. Since U is second countable, the Borel σ -algebra $\mathcal{B}(C^\infty(U, \mathbb{R}^m) \times U)$ coincides with the product σ -algebra $\mathcal{B}(C^\infty(U, \mathbb{R}^m)) \otimes \mathcal{B}(U)$ (see 1.6(f)). Therefore the map

$$\gamma \times \text{id}_U : [0, 1] \times U \rightarrow C^\infty(U, \mathbb{R}^m) \times U, \quad (t, x) \mapsto (\gamma(t), x)$$

is Borel measurable. The evaluation map

$$\varepsilon: C^\infty(U, \mathbb{R}^m) \times U \rightarrow \mathbb{R}^m, \quad \varepsilon(f, x) := f(x)$$

is C^∞ (see, e.g., [25] or [38]), hence continuous and hence measurable. Thus $\hat{\gamma} = \varepsilon \circ (\gamma \times \text{id}_{\mathbb{R}^n})$ is measurable. As $(\text{id}_{[0,1]}, \zeta): [0, 1] \rightarrow [0, 1] \times U$, $t \mapsto (t, \zeta(t))$ is measurable, also the composition $(\hat{\gamma})_*(\zeta) = \hat{\gamma} \circ (\text{id}_{[0,1]}, \zeta)$ is measurable. On \mathbb{R}^m , we use the maximum-norm $\|\cdot\|_\infty$, giving rise to a continuous norm $q := \|\cdot\|_{\bar{V}, \|\cdot\|_\infty}$ on $C^\infty(U, \mathbb{R}^m)$, which in turn gives rise to a continuous seminorm $\|\cdot\|_{\mathcal{L}^1, q}$ on $\mathcal{L}^1([0, 1], C^\infty(U, \mathbb{R}^m))$. Now

$$\|(\hat{\gamma})_*(\zeta)(t)\|_\infty = \|\gamma(t)(\zeta(t))\|_\infty \leq \|\gamma(t)\|_{\bar{V}, \|\cdot\|_\infty} = q(\gamma(t)), \quad (72)$$

whence $\int_0^1 \|(\hat{\gamma})_*(\zeta)(t)\|_\infty dt \leq \int_0^1 q(\gamma(t)) dt = \|\gamma\|_{\mathcal{L}^1, q} < \infty$ and hence $(\hat{\gamma})_*(\zeta) \in \mathcal{L}^1([0, 1], \mathbb{R}^m)$, with $\|(\hat{\gamma})_*(\zeta)\|_{\mathcal{L}^1, \|\cdot\|_\infty} \leq \|\gamma\|_{\mathcal{L}^1, q}$. \square

With notation as in the preceding lemma, we have:

Lemma 9.14 *For each $m \in \mathbb{N}$, the map*

$$\Phi_m: L^1([0, 1], C^\infty(U, \mathbb{R}^m)) \times C([0, 1], V) \rightarrow L^1([0, 1], \mathbb{R}^m),$$

$\Phi_m([\gamma], \zeta) := [(\hat{\gamma})_*(\zeta)]$, *is smooth.*

Proof. Since $\Phi_m(\gamma, \zeta)$ is linear in γ , it suffices to show that Φ_m is $C^{0,\infty}$ (see 1.56 (b) and (a)). We show by induction that Φ_m is $C^{0,k}$ for each $k \in \mathbb{N}_0$. Let $k = 0$ first; we have to show that Φ_m is continuous. From the preceding proof, we know that

$$\|\Phi_m(\gamma, \zeta)\|_{L^1, \|\cdot\|_\infty} \leq \|\gamma\|_{L^1, q}$$

with $q := \|\cdot\|_{\bar{V}, \|\cdot\|_\infty}$. Consider the map $D: C^\infty(U, \mathbb{R}^m) \rightarrow C^\infty(U, \mathbb{R}^{m \times n})$ such that $D(f)(x) := f'(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{m \times n}$ is the Jacobi matrix of f at x . Then D is continuous linear, entailing that $p := \|\cdot\|_{\bar{V}, \|\cdot\|_\infty} \circ D$ is a continuous seminorm on $C^\infty(U, \mathbb{R}^m)$. Thus $p(f) = \sup_{x \in \bar{V}} \|f'(x)\|_{op}$ for $f \in C^\infty(U, \mathbb{R}^m)$. Let $\tilde{\gamma} \in \mathcal{L}^1([0, 1], C^\infty(U, \mathbb{R}^m))$ and $\gamma := [\tilde{\gamma}] \in L^1([0, 1], C^\infty(U, \mathbb{R}^m))$. We have for all $\eta, \eta_1 \in C([0, 1], V)$ and $t \in [0, 1]$

$$\tilde{\gamma}(t)(\eta(t)) - \tilde{\gamma}(t)(\eta_1(t)) = \int_0^1 (D(\tilde{\gamma}(t))(\eta_1(t) + s(\eta(t) - \eta_1(t))).(\eta(t) - \eta_1(t)) ds \quad (73)$$

and thus

$$\begin{aligned}
& \|\tilde{\gamma}(t)(\eta(t)) - \tilde{\gamma}(t)(\eta_1(t))\|_\infty \\
& \leq \int_0^1 \|(D(\tilde{\gamma}(t))(\eta_1(t) + s(\eta(t) - \eta_1(t)))) \cdot (\eta(t) - \eta_1(t))\|_\infty ds \\
& \leq \|D(\tilde{\gamma}(t))\|_{\overline{V}, \|\cdot\|_{op}} \|\eta - \eta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty} \\
& = p(\tilde{\gamma}(t)) \|\eta - \eta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty}.
\end{aligned} \tag{74}$$

Integrating over t , we deduce that

$$\|\Phi_m(\gamma)(\eta) - \Phi_m(\gamma)(\eta_1)\|_{L^1, \|\cdot\|_\infty} \leq \|\gamma\|_{L^1, p} \|\eta - \eta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty}.$$

As a consequence,

$$\begin{aligned}
& \|\Phi_m(\gamma, \zeta) - \Phi_m(\gamma_1, \zeta_1)\|_{L^1, \|\cdot\|_\infty} \\
& \leq \|\Phi_m(\gamma, \zeta) - \Phi_m(\gamma, \zeta_1)\|_{L^1, \|\cdot\|_\infty} + \|\Phi_m(\gamma, \zeta_1) - \Phi_m(\gamma_1, \zeta_1)\|_{L^1, \|\cdot\|_\infty} \\
& \leq \|\gamma\|_{L^1, p} \|\eta - \eta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty} + \|\gamma - \gamma_1\|_{L^1, q}
\end{aligned}$$

for all $\gamma_1 \in L^1([0, 1], C^\infty(U, \mathbb{R}^m))$ and γ, η, η_1 as before, which can be made arbitrarily small for γ_1 close to γ and η_1 close to η . Thus Φ_m is continuous at each (γ, η) and thus Φ_m is continuous.

Let $k \in \mathbb{N}$ now and assume that Φ_m is $C^{0, k-1}$ for each $m \in \mathbb{N}$. Let $\gamma = [\tilde{\gamma}] \in L^1([0, 1], C^\infty(U, \mathbb{R}^m))$ with $\tilde{\gamma} \in \mathcal{L}^1([0, 1], C^\infty(U, \mathbb{R}^m))$; let $\eta \in C([0, 1], V)$ and $\eta_1 \in C([0, 1], \mathbb{R}^n)$. Since $\eta([0, 1])$ is a compact subset of the open set V and $\eta_1([0, 1])$ is compact, there is $\delta > 0$ such that

$$\eta(t) + r\eta_1(t) \in V \quad \text{for all } r \in [-\delta, \delta] \text{ and } t \in [0, 1].$$

To calculate $d_2\Phi_m(\gamma, \eta; \eta_1)$, we consider the corresponding directional difference quotients first. For $t \in [0, 1]$ and $\tau \in]-\delta, \delta[\setminus \{0\}$, we have

$$\frac{\tilde{\gamma}(t)(\eta(t) + \tau\eta_1(t)) - \tilde{\gamma}(t)(\eta(t))}{\tau} = \int_0^1 (D(\tilde{\gamma}(t))(\eta(t) + s\tau\eta_1(t)) \cdot \eta_1(t)) ds \tag{75}$$

by (73). The map

$$\alpha: L^1([0, 1], \mathbb{R}^{m \times n}) \rightarrow L^1([0, 1], \mathbb{R}^m), \quad \alpha([f]) := [s \mapsto f(s)\eta_1(s)]$$

for $f \in \mathcal{L}^1([0, 1], \mathbb{R}^{m \times n})$, $s \in [0, 1]$ (given pointwise by multiplication of matrices and vectors) is linear and continuous, with $\|\alpha\|_{op} \leq \|\eta_1\|_{\mathcal{L}^\infty, \|\cdot\|_\infty}$. We

abbreviate $\bar{\gamma} := D \circ \tilde{\gamma}$ and identify $\mathbb{R}^{m \times n}$ with \mathbb{R}^{mn} . Then the mapping $h: \mathbb{R} \times [0, 1] \rightarrow L^1([0, 1], \mathbb{R}^m)$,

$$h(\tau, s) := \alpha([\widehat{\gamma}]_*(\eta + s\tau\eta_1)) = \alpha(\Phi_{mn}([\bar{\gamma}], \eta + s\tau\eta_1))$$

is continuous, by induction, and we record that

$$h(0, s) = \alpha(\Phi_{mn}([\bar{\gamma}], \eta)) = \Phi_{mn}([\bar{\gamma}], \eta)\eta_1$$

is independent of $s \in [0, 1]$. Now the theorem on parameter-dependent integrals (see 1.18) shows that

$$g:]-\delta, \delta[\rightarrow L^1([0, 1], \mathbb{R}^m), \quad g(\tau) := \int_0^1 h(\tau, s) ds$$

is continuous. Then

$$g(\tau) = \frac{\Phi_m(\gamma, \eta + \tau\eta_1) - \Phi_m(\gamma, \eta)}{\tau} \quad \text{for all } \tau \in]-\delta, \delta[\setminus \{0\};$$

this can be shown as in the proof of Lemma 9.9 (using the next lemma). Now the continuity of g implies that the limit as $\tau \rightarrow 0$ exists; we have

$$\begin{aligned} d_2\Phi_m(\gamma, \eta; \eta_1) &= \lim_{\tau \rightarrow 0} \frac{\Phi_m(\gamma, \eta + \tau\eta_1) - \Phi_m(\gamma, \eta)}{\tau} \\ &= \lim_{\tau \rightarrow 0} g(\tau) = g(0) = \int_0^1 h(0, s) ds = \Phi_{mn}([\bar{\gamma}], \eta)\eta_1. \end{aligned}$$

The map

$$\beta: L^1([0, 1], \mathbb{R}^{m \times n}) \times C([0, 1], \mathbb{R}^n) \rightarrow L^1([0, 1], \mathbb{R}^m), \quad \beta([f], g) := [t \mapsto f(t)g(t)]$$

given by pointwise multiplication of matrices and vectors is continuous bilinear with $\|\beta\|_{\text{op}} \leq 1$, and hence smooth. By the preceding, we have

$$d_2\Phi_m(\gamma, \eta; \eta_1) = \beta(\Phi_{mn}([\bar{\gamma}], \eta), \eta_1). \quad (76)$$

The map

$$L^1([0, 1], D): L^1([0, 1], C^\infty(U, \mathbb{R}^m)) \rightarrow L^1([0, 1], C^\infty(U, \mathbb{R}^{m \times n}))$$

sending $\gamma = [\tilde{\gamma}]$ to $[\bar{\gamma}] = [D \circ \tilde{\gamma}]$ is continuous linear. The map Φ_{mn} is $C^{0, k-1}$ by induction. Hence also $(\gamma, \eta) \mapsto \Phi_{mn}([\bar{\gamma}], \eta)$ is $C^{0, k-1}$ (see 1.56 (c)). Looking at the right hand side of (70), we deduce with 1.56 (d) that $d_2\Phi_m$ is $C^{0, k-1, \infty}$ as a function of (γ, η, η_1) and hence (by 1.56 (e)) $C^{0, k-1}$ as a function of $(\gamma, (\eta, \eta_1))$. Hence Φ_m is $C^{0, k}$, by 1.56 (f). \square

The following analogue of Lemma 9.10 was used.

Lemma 9.15 *The function $f: [0, 1]^2 \rightarrow \mathbb{R}$,*

$$(t, s) \mapsto \lambda(D(\tilde{\gamma}(t))(\eta(t) + s\tau\eta_1(t))\eta_1(t))\theta(t)$$

is in $\mathcal{L}^1([0, 1]^2, \mathbb{R})$ with respect to Lebesgue-Borel measure on $[0, 1]^2$.

Proof. To see that f is measurable, write $\tilde{\gamma}(t) = (\tilde{\gamma}_1(t), \dots, \tilde{\gamma}_m(t))$ (identifying $C^\infty(U, \mathbb{R}^m)$ with $C^\infty(U, \mathbb{R})^m$). Then $t \mapsto \frac{\partial \tilde{\gamma}_i(t)}{\partial x_j}$ is an element of $\mathcal{L}^1([0, 1], C^\infty(U, \mathbb{R}))$ for each $j \in \{1, \dots, n\}$. Write $\eta_1 = (\eta_{1,1}, \dots, \eta_{1,n})$ with continuous functions $\eta_{1,j}: [0, 1] \rightarrow \mathbb{R}$ for $j \in \{1, \dots, n\}$. There are $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that $\lambda(x_1, \dots, x_m) = \lambda_1 x_1 + \dots + \lambda_m x_m$. The evaluation map

$$\varepsilon: C([0, 1], \mathbb{R}^n) \times [0, 1] \rightarrow \mathbb{R}^n, \quad (\kappa, t) \mapsto \kappa(t)$$

is continuous and hence measurable. The map $[0, 1]^2 \rightarrow C([0, 1], \mathbb{R}^n) \times [0, 1]$, $(s, t) \mapsto (\eta + s\tau\eta_1, t)$ is continuous and hence measurable. Now the formula

$$f(t, s) = \sum_{i=1}^m \sum_{j=1}^n \lambda_i \frac{\partial \tilde{\gamma}_i(t)}{\partial x_j} (\varepsilon(\eta + s\tau\eta_1, t)) \eta_{1,j}(t) \theta(t)$$

shows that f is measurable, being a sum of products of measurable real-valued functions. Using Fubini's theorem for non-negative measurable functions on $[0, 1]^2$, we find that

$$\begin{aligned} & \int_{[0,1]^2} |f(t, s)| d\lambda_2(t, s) \\ &= \int_0^1 \int_0^1 |f(t, s)| ds dt \\ &\leq \|\theta\|_{\mathcal{L}^\infty} \sum_{i=1}^m \sum_{j=1}^n \|\eta_{1,j}\|_{\mathcal{L}^\infty} |\lambda_i| \int_0^1 \int_0^1 \underbrace{\left| \frac{\partial \tilde{\gamma}_i(t)}{\partial x_j} (\varepsilon(\eta + s\tau\eta_1, t)) \right|}_{\leq \left\| \frac{\partial \tilde{\gamma}_i(t)}{\partial x_j} \right\|_{\nabla, |\cdot|}} ds dt \\ &\leq \int_0^1 \left\| \frac{\partial \tilde{\gamma}_i}{\partial x_j} \right\|_{\nabla, |\cdot|} dt = \|\partial/\partial x_j \circ \tilde{\gamma}_i\|_{\mathcal{L}^1, p} < \infty \end{aligned}$$

with $p := \|\cdot\|_{\nabla, |\cdot|}$. □

9.16 For $z \in \mathbb{Z}^n$ and $r > 0$, let $B_r(z) \subseteq \mathbb{R}^n$ be the ball with respect to $\|\cdot\|_\infty$. For $z \in \mathbb{Z}^n$, the balls $B_1(z)$ form a locally finite open cover of \mathbb{R}^n by relatively compact open sets $B_1(z)$; likewise for $B_3(z)$. Hence

$$\rho_1: C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \bigoplus_{z \in \mathbb{Z}^n} C^\infty(B_1(0), \mathbb{R}^n), \quad \gamma \mapsto (f|_{B_1(z)})_{z \in \mathbb{Z}^n}$$

and the corresponding maps

$$\rho_3: C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \bigoplus_{z \in \mathbb{Z}^n} C^\infty(B_3(0), \mathbb{R}^n)$$

and $\rho_4: C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \bigoplus_{z \in \mathbb{Z}^n} C^\infty(B_4(0), \mathbb{R}^n)$ are linear topological embeddings with closed (and complemented) image (see, e.g., [25]). Explicitly, $\text{im}(\rho_1)$ is the set of all $\{(\gamma_z)_{z \in \mathbb{Z}^n} \in \bigoplus_{z \in \mathbb{Z}^n} C^\infty(B_1(z), \mathbb{R}^n)$ such that

$$(\forall z, w \in \mathbb{Z}^n)(\forall x \in B_1(z) \cap B_1(w)) \quad \gamma_z(x) = \gamma_w(x). \quad (77)$$

As a consequence, also the maps

$$\begin{aligned} R_3 &:= L^1([0, 1], \rho_3): L^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)) \rightarrow L^1\left([0, 1], \bigoplus_{z \in \mathbb{Z}^n} C^\infty(B_3(z), \mathbb{R}^n)\right) \\ &\cong \bigoplus_{z \in \mathbb{Z}^n} L^1([0, 1], C^\infty(B_3(z), \mathbb{R}^n)), \end{aligned} \quad (78)$$

$$R_4 := L^1([0, 1], \rho_4): L^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)) \rightarrow \bigoplus_{z \in \mathbb{Z}^n} L^1([0, 1], C^\infty(B_4(z), \mathbb{R}^n))$$

and

$$\begin{aligned} R_1 &:= C([0, 1], \rho_1): C([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)) \rightarrow C\left([0, 1], \bigoplus_{z \in \mathbb{Z}^n} C^\infty(B_1(z), \mathbb{R}^n)\right) \\ &\cong \bigoplus_{z \in \mathbb{Z}^n} C([0, 1], C^\infty(B_1(z), \mathbb{R}^n)) \end{aligned} \quad (79)$$

are linear topological embeddings with closed image (where we use Mujica's Theorem and its analogue for Lebesgue spaces discussed above to rewrite the spaces as direct sums).

Since ρ_1 is a topological embedding and $\rho_1(0) = 0$, there exist open 0-neighbourhoods $W_z \subseteq C^\infty(B_1(z), \mathbb{R}^n)$ such that

$$(\rho_1)^{-1}\left(\bigoplus_{z \in \mathbb{Z}^n} W_z\right) \subseteq \Omega. \quad (80)$$

9.17 Consider the continuous seminorm $p_z := \|\cdot\|_{\overline{B}_2(z), \|\cdot\|_{op}} \circ D + \|\cdot\|_{\overline{B}_2(z), \|\cdot\|_\infty}$ on $C^\infty(B_3(z), \mathbb{R}^n)$, where $D: C^\infty(B_3(z), \mathbb{R}^n) \rightarrow C^\infty(B_3(z), \mathbb{R}^{n \times n})$, $f \mapsto f'$; thus

$$p_z(f) = \sup_{x \in \overline{B}_2(z)} (\|f'(x)\|_{op} + \|f(x)\|_\infty) \quad \text{for } f \in C^\infty(B_3(z), \mathbb{R}^n).$$

Fix $L \in]0, 1[$. Then

$$Q_z := \{\gamma \in L^1([0, 1], C^\infty(B_3(0), \mathbb{R}^n)) : \|\gamma\|_{L^1, p_z} < L\}$$

is an open 0-neighbourhood in $L^1([0, 1], C^\infty(B_3(z), \mathbb{R}^n))$. We define a map $\Psi_z: Q_z \times B_1(z) \times C([0, 1], B_2(z)) \rightarrow C([0, 1], B_2(z))$ via

$$\Psi_z([\gamma], x, \kappa)(t) := x + \int_0^t \gamma(s)(\kappa(s)) ds$$

for $[\gamma] \in Q_z$ with $\gamma \in \mathcal{L}^1([0, 1], C^\infty(B_3(z), \mathbb{R}^n))$, $x \in B_1(z)$, $\kappa \in C([0, 1], B_2(z))$ and $t \in [0, 1]$.

Lemma 9.18 *The map $\Psi_z: Q_z \times B_1(z) \times C([0, 1], B_2(z)) \rightarrow C([0, 1], B_2(z))$ is smooth and defines a uniform family of contractions in the final variable, in the sense that*

$$\text{Lip}(\Psi_z([\gamma], x, \bullet)) \leq L$$

for all $[\gamma] \in Q_z$ and all $x \in B_1(z)$.

Proof. For $x \in \mathbb{R}^n$, let $c_x: [0, 1] \rightarrow \mathbb{R}^n$ be the constant function $t \mapsto x$. The map $\mathbb{R}^n \rightarrow C([0, 1], \mathbb{R}^n)$, $x \mapsto c_x$ is continuous linear and hence smooth. Moreover, the operator

$$J: L^1([0, 1], \mathbb{R}^n) \rightarrow C([0, 1], \mathbb{R}^n)$$

determined by $J([f])(t) := \int_0^t f(s) ds$ is continuous and linear and hence smooth. Consider the mapping

$$\Phi_z: L^1([0, 1], C^\infty(B_3(z), \mathbb{R}^m)) \times C([0, 1], B_2(0)) \rightarrow L^1([0, 1], \mathbb{R}^n),$$

$\Phi_z([\gamma], \zeta) := [(\hat{\gamma})_*(\zeta)]$, which is smooth by Lemma 9.14). Now the formula

$$\Psi_z([\gamma], x, \kappa) = c_x + J(\Phi_z([\gamma], \kappa))$$

shows that Ψ_z is smooth. Given $\eta, \eta_1 \in C([0, 1], B_2(0))$, we deduce from (74) that

$$\begin{aligned} \|\Psi_z([\gamma], x, \eta) - \Psi_z([\gamma], x, \eta_1)\|_\infty &= \sup_{t \in [0, 1]} \left\| \int_0^t \gamma(s)(\eta(s)) - \gamma(s)(\eta_1(s)) ds \right\|_\infty \\ &\leq \sup_{t \in [0, 1]} \int_0^t \underbrace{\|\gamma(s)(\eta(s)) - \gamma(s)(\eta_1(s))\|_\infty}_{\leq p(\gamma(t))\|\eta - \eta_1\|_{L^\infty}} ds \\ &\leq \|\gamma\|_{\mathcal{L}^1, p_z} \|\eta - \eta_1\|_{L^\infty} \leq L \|\eta - \eta_1\|_{L^\infty} \end{aligned}$$

with p_z as in 9.16. It only remains to observe that $\Psi_z([\gamma], x, \eta)(t) \in B_2(z)$ always since $\|\int_0^t \gamma(s)(\eta(s)) ds\|_\infty \leq \int_0^t \|\gamma(s)(\eta(s))\|_\infty ds \leq \int_0^t \|\gamma(s)\|_{\overline{B}_2(z), \|\cdot\|_\infty} ds \leq \int_0^t \|\gamma(s)\|_{\overline{B}_2(z), \|\cdot\|_\infty} ds \leq \int_0^t p_z(\gamma(s)) ds \leq \|\gamma\|_{\mathcal{L}^1, p_z} \leq 1$. \square

9.19 If $([\gamma], x) \in Q_z \times B_1(z)$, we have $x \in \overline{B}_r(z)$ for some $r \in]0, 1[$ and $\Psi_z([\gamma], x, \bullet)$ restricts to a contraction

$$C([0, 1], \overline{B}_{1+r}(z)) \rightarrow C([0, 1], \overline{B}_{1+r}(z))$$

of the complete metric space $C([0, 1], \overline{B}_{1+r}(z)) = \{\zeta \in C([0, 1], \mathbb{R}^n) : \|\zeta\|_\infty \leq 1+r\}$. The latter has a unique fixed point $\zeta_{[\gamma], x}$ by Banach's Contraction Principle, which then also is the unique fixed point of the contraction $\Psi_z([\gamma], x, \bullet)$ of $C([0, 1], B_2(0))$. Thus

$$\Psi_z([\gamma], x, \zeta_{[\gamma], x}) = \zeta_{[\gamma], x}. \quad (81)$$

Since Ψ_z is smooth, Lemma 6.2 shows that also the map

$$Q_z \times B_1(0) \rightarrow C([0, 1], B_2(0)), \quad ([\gamma], x) \mapsto \zeta_{[\gamma], x}$$

is smooth. Define $F_z([\gamma])(t)(x) := \zeta_{[\gamma], x}(t)$ for $[\gamma] \in Q_z$, $x \in B_1(0)$ and $t \in [0, 1]$. Using the exponential laws from [2], we deduce:

- (a) $F_z([\gamma])(t) \in C^\infty(B_1(z), \mathbb{R}^n)$ for all $[\gamma] \in Q_z$ and $t \in [0, 1]$;
- (b) $F_z([\gamma]) \in C([0, 1], C^\infty(B_1(z), \mathbb{R}^n))$ for all $[\gamma] \in Q_z$;
- (c) $F_z : Q_z \rightarrow C([0, 1], C^\infty(B_1(z), \mathbb{R}^n))$ is smooth.

9.20 As a consequence, also the map

$$E_z: Q_z \rightarrow C([0, 1], C^\infty(B_1(z), \mathbb{R}^n)), \quad [\gamma] \mapsto F_z([\gamma]) - I$$

is smooth, where $I: [0, 1] \rightarrow C^\infty(B_1(z), \mathbb{R}^n)$ is the constant function $t \mapsto \text{id}_{B_1(z)}$. If $\gamma = 0$, then $\zeta_{[\gamma], x}$ is the fixed point of the map $\Psi_z([\gamma], x, \bullet)$ determined by

$$\Psi_z([0], x, \kappa)(t) = x.$$

Hence $\zeta_{0, x} = c_x$. As a consequence,

$$E_z(0)(t) = 0$$

and thus $E_z(0) = 0$. Since E_z is continuous, there is an open 0-neighbourhood $P_z \subseteq Q_z$ such that

$$E_z(P_z) \subseteq W_z$$

(where W_z is as in (80)).

9.21 Consider $D: C^\infty(B_4(z), \mathbb{R}^n) \rightarrow C^\infty(B_4(z), \mathbb{R}^{n \times n})$, $f \mapsto f'$ and the continuous seminorm $q_z := \|\cdot\|_{\overline{B}_3(z), \|\cdot\|_{op}} \circ D + \|\cdot\|_{\overline{B}_3(0), \|\cdot\|_\infty}$ on $C^\infty(B_4(z), \mathbb{R}^n)$ for $z \in \mathbb{Z}^n$; thus

$$q_z(f) = \sup_{x \in \overline{B}_3(z)} (\|f'(x)\|_{op} + \|f(x)\|_\infty) \quad \text{for } f \in C^\infty(B_4(z), \mathbb{R}^n).$$

Then

$$S_z := \{\gamma \in L^1([0, 1], C^\infty(B_4(0), \mathbb{R}^n)) : \|\gamma\|_{L^1, q_z} < L\}$$

is an open 0-neighbourhood in $L^1([0, 1], C^\infty(B_4(z), \mathbb{R}^n))$. We define a map $\Theta_z: S_z \times B_2(z) \times C([0, 1], B_3(z)) \rightarrow C([0, 1], B_3(z))$ via

$$\Theta_z([\gamma], x, \kappa)(t) := x + \int_0^t \gamma(s)(\kappa(s)) ds$$

for $[\gamma] \in S_z$ with $\gamma \in \mathcal{L}^1([0, 1], C^\infty(B_4(z), \mathbb{R}^n))$, $x \in B_2(z)$, $\kappa \in C([0, 1], B_3(z))$ and $t \in [0, 1]$.

The following lemma can be shown like Lemma 9.18.

Lemma 9.22 *For all $z \in \mathbb{Z}^n$, $[\gamma] \in S_z$ and $x \in B_2(z)$, the map*

$$\Theta_z([\gamma], x, \bullet): C([0, 1], B_3(z)) \rightarrow C([0, 1], B_3(z)), \quad \kappa \mapsto \Theta_z([\gamma], x, \kappa)$$

is a contraction, with $\text{Lip}(\Theta_z([\gamma], x, \bullet)) \leq L$.

□

9.23 Now consider the open 0-neighbourhood

$$\mathcal{U} := R_3^{-1} \left(\bigoplus_{z \in \mathbb{Z}^n} P_z \right) \cap R_4^{-1} \left(\bigoplus_{z \in \mathbb{Z}^n} S_z \right)$$

in $L^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$. Since E_z is smooth for each $z \in \mathbb{Z}^n$ and $E_z(0) = 0$, also the map

$$\bigoplus_{z \in \mathbb{Z}^n} E_z : \bigoplus_{z \in \mathbb{Z}^n} Q_z \rightarrow \bigoplus_{z \in \mathbb{Z}^n} C([0, 1], C^\infty(B_1(z), \mathbb{R}^n)), \quad (\gamma_z)_{z \in \mathbb{Z}^n} \mapsto (E_z(\gamma_z))_{z \in \mathbb{Z}^n}$$

is smooth (see [24]). We claim that

$$\left(\bigoplus_{z \in \mathbb{Z}^n} E_z \right) (R_3([\gamma])) \in R_1(C([0, 1], \Omega)) \quad (82)$$

for each $[\gamma] \in \mathcal{U}$. If this is true, then $(\bigoplus_{z \in \mathbb{Z}^n} E_z) \circ R_3|_{\mathcal{U}}$ is smooth also as a map $\mathcal{U} \rightarrow \text{im}(R_1)$ (since $\text{im}(R_1)$ is a closed vector subspace and thus [5, Lemma 10.1] applies). As a consequence, also the map

$$E := R_1^{-1} \circ \left(\bigoplus_{z \in \mathbb{Z}^n} E_z \right) \circ R_3|_{\mathcal{U}} : \mathcal{U} \rightarrow C([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$$

is smooth. Let $[\gamma] \in \mathcal{U}$ with $\gamma \in \mathcal{L}^1([0, 1], C_c^\infty(\mathbb{R}, \mathbb{R}^n))$. If $x \in \mathbb{R}^n$, we can find $w \in \mathbb{Z}^n$ such that $x \in B_1(w)$. Let us write $h : [0, 1] \rightarrow C^\infty(B_1(w), \mathbb{R}^N)$ for the w -component of

$$R_1(E([\gamma])) = \left(\bigoplus_{z \in \mathbb{Z}^n} E_z \right) (R_3([\gamma])).$$

Then $h = E_w([s \mapsto \gamma(s)|_{B_3(w)}])$ and

$$h(t) = E([\gamma])(t)|_{B_1(w)},$$

entailing that $E([\gamma])(t)(x) = h(t)(x) = E_w([s \mapsto \gamma(s)|_{B_3(w)}])(t)$ for all $t \in [0, 1]$. Thus $[0, 1] \rightarrow E([\gamma])(t)(x)$ is a Carathéodory solution to

$$y'(t) = \gamma(t)(y(t)), \quad y(0) = x$$

and thus (64) is satisfied by the continuous function $\eta := E([\gamma]) : [0, 1] \rightarrow \Omega \subseteq C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. As a consequence, $\eta = \text{Evol}_\Omega^r([\gamma])$ is the right evolution

of $[\gamma]$ (see 9.6). Since L^1 has the subdivision property by Lemma 5.26, we deduce with (d) \Rightarrow (c) in Proposition 5.25 that Evol^r exists on all of $L^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$. Since $\text{Evol}^r|_{\mathcal{U}} = E$ is smooth, (b) \Rightarrow (a) in Proposition 5.25 now shows that the Lie group $(\Omega, *)$ (and hence also $\text{Diff}_c(\mathbb{R}^n)$) is L^1 -regular.

9.24 To complete the proof of L^1 -regularity for $\text{Diff}_c(\mathbb{R}^n)$, it only remains to prove (82). To this end, let $[\gamma] \in \mathcal{U}$ with $\gamma \in \mathcal{L}^1([0, 1], C_c^\infty(\mathbb{R}^n, \mathbb{R}^n))$. For $z \in \mathbb{Z}^n$, define $\gamma_z \in \mathcal{L}^1([0, 1], C^\infty(B_3(z), \mathbb{R}^n))$ via $\gamma_z(t) := \gamma(t)|_{B_3(z)}$. Define $\tilde{\gamma}_z \in \mathcal{L}^1([0, 1], C^\infty(B_4(0), \mathbb{R}^n))$ via $\tilde{\gamma}_z(t) := \gamma(t)|_{B_4(z)}$. Let $z, w \in \mathbb{Z}^n$ and $x \in B_1(z) \cap B_1(w)$. Let us write $\zeta_{[\gamma_z], x}^z: [0, 1] \rightarrow B_2(z)$ for the unique fixed point of $\Psi_z([\gamma_z], x, \bullet)$ and $\zeta_{[\gamma_w], x}^w: [0, 1] \rightarrow B_2(w)$ for the unique fixed point of $\Psi_w([\gamma_w], x, \bullet)$. Thus $\zeta_{[\gamma_z], x}^z$ is a Carathéodory solution to

$$y'(t) = x + \int_0^t \gamma(t)|_{B_3(z)}(y(t)) dt, \quad y(0) = x. \quad (83)$$

and $\zeta_{[\gamma_w], x}^w$ is a Carathéodory solution to

$$y'(t) = x + \int_0^t \gamma(t)|_{B_3(w)}(y(t)) dt, \quad y(0) = x. \quad (84)$$

Since $B_2(z)$ and $B_2(w)$ are subsets of $B_3(z)$, both $\zeta_{[\gamma_z], x}^z$ and $\zeta_{[\gamma_w], x}^w$ can be considered as elements of $C([0, 1], B_3(z))$. Since both $B_3(z)$ and $B_3(w)$ are subsets of $B_4(z)$, we deduce from (83) and (84) that both $\zeta_{[\gamma_z], x}^z$ and $\zeta_{[\gamma_w], x}^w$ are Carathéodory solutions to

$$y'(t) = x + \int_0^t \gamma(t)|_{B_4(w)}(y(t)) dt, \quad y(0) = x$$

and hence fixed points of $\Theta_z([\tilde{\gamma}_z], x, \bullet)$. As the contraction

$$\Theta_z([\tilde{\gamma}_z], x, \bullet): C([0, 1], B_3(z)) \rightarrow C([0, 1], B_3(z))$$

has at most one fixed point, we deduce that $\zeta_{[\gamma_z], x}^z = \zeta_{[\gamma_w], x}^w$. Thus

$$E_z([\gamma_z])(t)(x) = \zeta_{[\gamma_z], x}^z(t) - x = \zeta_{[\gamma_w], x}^w(t) - x = E_w([\gamma_w])(t)(x)$$

for each $t \in [0, 1]$. Hence

$$(E_z([\gamma_z])(t))_{z \in \mathbb{Z}^n} \in \text{im}(\rho_1)$$

for each $t \in [0, 1]$, using (77). As a consequence,

$$(E_z([\gamma_z]))_{z \in \mathbb{Z}^n} = (\oplus_{z \in \mathbb{Z}^n} E_z)(R_3([\gamma])) \in \text{im}(R_1).$$

Since $(E_z([\gamma_z])(t))_{z \in \mathbb{Z}^n} \in W_z$, we deduce that $\rho_1^{-1}((E_z([\gamma_z])(t))_{z \in \mathbb{Z}^n}) \in \Omega$ for each t (see (80)), whence $(E_z([\gamma_z]))_{z \in \mathbb{Z}^n} = (\oplus_{z \in \mathbb{Z}^n} E_z)(R_3([\gamma])) \in R_1(\Omega)$. Thus (82) holds and the proof for L^1 -regularity of $\text{Diff}_c(\mathbb{R}^n)$ is complete.

10 Local L^1 -Lipschitz condition and uniqueness for initial value problems

It is well known from the classical Picard-Lindelöf Theorem that solutions to initial value problems in normed spaces are unique if the right hand side of the differential equation is continuous and satisfies a local Lipschitz condition. As a preparation for the discussion of L^1 -regularity for $\text{Diff}_c(M)$, we now describe weaker conditions ensuring uniqueness.³² For differential equations on subsets of finite-dimensional spaces, similar conditions have been used e.g. in [68, Appendix C.3, Theorem 54].

Definition 10.1 Let $(E, \|\cdot\|)$ be a normed space, $U \subseteq E$ be a subset and $a < b$ be real numbers. We say that a function $f: [a, b] \times U \rightarrow E$ satisfies a (global) L^1 -Lipschitz condition if there exists a measurable function $g: [a, b] \rightarrow [0, \infty]$ with $L := \int_a^b g(t) d\lambda_1(t) < \infty$ such that

$$\text{Lip}(f(t, \bullet)) \leq g(t) \quad \text{for all } t \in [a, b].$$

Remark 10.2 (a) Here $\text{Lip}(f(t, \bullet)) \in [0, \infty]$ means the infimum of all Lipschitz constants for the mapping $f_t := f(t, \bullet): U \rightarrow E$, $y \mapsto f(t, y)$.

(b) If the function $h: [a, b] \rightarrow [0, \infty]$, $t \mapsto \text{Lip}(f(t, \bullet))$ is measurable, then g as required in Definition 10.1 exists if and only if $g := h$ can be chosen there (i.e., if and only if h is integrable). In all of our applications, h is measurable, but we do not need this requirement to formulate Definition 10.1.

³²The alert reader may notice that the condition was already satisfied in the preceding section. However, making it explicit would not much shorten the proof, as we used Banach's Contraction Theorem anyway, and could exploit its uniqueness assertion.

Definition 10.3 Let M be a C^1 -manifold modelled on a normed space $(E, \|\cdot\|)$, $J \subseteq \mathbb{R}$ be a non-degenerate interval and $f: J \times M \rightarrow TM$ be a function with $f(t, p) \in T_p(M)$ for all $(t, p) \in J \times M$. We say that f *satisfies a local L^1 -Lipschitz condition* if for all $t_0 \in J$ and $p \in M$, there exists a chart

$$\kappa: U_\kappa \rightarrow V_\kappa \subseteq E$$

of M with $p \in U_\kappa$ and a relatively open subinterval $[a, b] \subseteq J$ which is a neighbourhood of t_0 in J such that the map

$$f_\kappa: [a, b] \times V_\kappa \rightarrow E, \quad (t, y) \mapsto d\kappa(f(t, \kappa^{-1}(y))) \quad (85)$$

satisfies an L^1 -Lipschitz condition.

Remark 10.4 If $(E, \|\cdot\|)$ is a normed space and $g: W \rightarrow E$ a C^1 -map on an open subset $W \subseteq E$, then each $x \in W$ has an open neighbourhood $W_0 \subseteq W$ such that $\sup_{y \in W_0} \|dg(y, \cdot)\|_{op} < \infty$ (e.g., any W_0 such that $dg(W_0 \times B_r^E(0)) \subseteq B_1^E(0)$ for some $r > 0$, which exists by continuity of $dg: W \times E \rightarrow E$). Choosing g as the transition map (change of charts) from one chart to another, we deduce that f_κ will actually satisfy an L^1 -Lipschitz condition for *each* chart around p in the situation of Definition 10.3, for suitable $[a, b]$.

Proposition 10.5 (Uniqueness of solutions) *Let $a < b$ be real numbers, M be a C^1 -manifold modelled on a normed space $(E, \|\cdot\|)$ and*

$$f: [a, b] \times M \rightarrow TM$$

be a function with $f(t, p) \in T_p(M)$ for all $(t, p) \in [a, b] \times M$, which satisfies a local L^1 -Lipschitz condition. If $\gamma: [a, b] \rightarrow M$ and $\eta: [a, b] \rightarrow M$ are absolutely continuous curves which are Carathéodory solutions to

$$y' = f(t, y)$$

and satisfy $\gamma(t_0) = \eta(t_0)$ for some $t_0 \in [a, b]$, then $\gamma = \eta$.

Proof. We show that $\gamma|_{[t_0, b]} = \eta|_{[t_0, b]}$; the proof that $\gamma|_{[a, t_0]} = \eta|_{[a, t_0]}$ is similar. We may therefore assume now that $t_0 = a$. The set

$$A := \{t \in [a, b]: \gamma(t) = \eta(t)\}$$

is closed in $[a, b]$ as γ and η are continuous and M is Hausdorff. Moreover, A is non-empty as $t_0 \in A$. If we can show that A is also open in $[a, b]$, then $A = [a, b]$ will follow from the connectedness of $[a, b]$ (completing the proof). Let $\tau \in A$. By Definition 10.3, there exist $\alpha < \beta$ in $[a, b]$ such that $[\alpha, \beta]$ is a neighbourhood of τ in $[a, b]$ and a chart $\kappa: U_\tau \rightarrow V_\tau \subseteq E$ such that

$$\gamma(\tau) = \eta(\tau) \in U_\kappa$$

and the map $f_\kappa: [\alpha, \beta] \times V_\kappa \rightarrow E$ (analogous to (85)) satisfies an L^1 -Lipschitz condition. Let $g: [\alpha, \beta] \rightarrow [0, \infty]$ be an integrable function with $\text{Lip}(f_\kappa(t, \bullet)) \leq g(t)$ for all $t \in [\alpha, \beta]$. Since

$$\lim_{r \rightarrow 0} \int_{[\alpha, \beta] \cap [\tau-r, \tau+r]} g(s) ds = 0,$$

after shrinking the neighbourhood $[\alpha, \beta]$ of τ , we may assume that

$$L := \int_\alpha^\beta g(s) ds < 1.$$

Abbreviate $\|\gamma - \eta\|_{[\alpha, \beta]} := \sup\{\|\gamma(t) - \eta(t)\| : t \in [\alpha, \beta]\}$. For each $t \in [\tau, \beta]$, we obtain

$$\begin{aligned} \|\gamma(t) - \eta(t)\| &= \left\| \int_\tau^t (f(s, \gamma(s)) - f(s, \eta(s))) ds \right\| \\ &\leq \int_\tau^t \underbrace{\|f(s, \gamma(s)) - f(s, \eta(s))\|}_{\leq \text{Lip}(f(s, \bullet)) \|\gamma(s) - \eta(s)\| \leq g(s) \|\gamma(s) - \eta(s)\|} ds \\ &\leq \int_\tau^t g(s) \|\gamma(s) - \eta(s)\| ds \leq L \|\gamma - \eta\|_{[\alpha, \beta]}. \end{aligned}$$

A similar argument shows that $\|\gamma(t) - \eta(t)\| \leq L \|\gamma - \eta\|_{[\alpha, \beta]}$ also for $t \in [\alpha, \tau]$. Hence $\|\gamma(t) - \eta(t)\| \leq L \|\gamma - \eta\|_{[\alpha, \beta]}$ for all $t \in [\alpha, \beta]$ and thus

$$\|\gamma - \eta\|_{[\alpha, \beta]} \leq L \|\gamma - \eta\|_{[\alpha, \beta]},$$

which is impossible unless $\|\gamma - \eta\|_{[\alpha, \beta]} = 0$ and thus $\gamma|_{[\alpha, \beta]} = \eta|_{[\alpha, \beta]}$. Thus $[\alpha, \beta] \subseteq A$, entailing that A is a neighbourhood of τ and hence open in $[a, b]$ (as $\tau \in A$ was arbitrary), which completes the proof. \square

Definition 10.6 Let M be a C^1 -manifold modelled on a normed space $(E, \|\cdot\|)$. Let t_0, T be real numbers and $f: [t_0, t_0 + T] \times M \rightarrow TM$ be a function with $f(t, p) \in T_p(M)$ for all $(t, p) \in [t_0, t_0 + T] \times M$, which satisfies a local L^1 -Lipschitz condition. If the initial value problem

$$y'(t) = f(t, y'(t)), \quad y(t_0) = p$$

has a (necessarily unique) solution $\gamma_p: [t_0, t_0 + T] \rightarrow M$ for each $p \in M$, then we say that f admits a global flow for initial time t_0 and write

$$\Phi_{t, t_0}^f(p) := \gamma_p(t)$$

for $t \in [t_0, t_0 + T]$ and $p \in M$. In this way, for each $t \in [t_0, t_0 + T]$ we obtain a mapping $\Phi_{t, t_0}^f: M \rightarrow M$.

11 L^1 -regularity of $\text{Diff}_c(M)$ and $\text{Diff}_K(M)$

Throughout this section, let M be a paracompact finite-dimensional smooth manifold and $K \subseteq M$ be a compact subset (starting with 11.3, we shall assume that M is σ -compact). Our goal is to see that $\text{Diff}_c(M)$ and $\text{Diff}_K(M)$ are L^1 -regular. For general information on the Lie group structure of $\text{Diff}_c(M)$, the reader is referred to [51], [22], and [38]. The Lie group $\text{Diff}_c(M)$ can also be regarded as a special case of the diffeomorphism groups of orbifolds discussed in [63].

11.1 Let g be a smooth Riemannian metric on M and $\exp: \mathcal{D} \rightarrow M$ be the Riemannian exponential function, defined on its maximal domain \mathcal{D} which is an open neighbourhood of the zero-section $M \rightarrow TM$, $p \mapsto 0_p \in T_p M$ in TM . Let $\pi_{TM}: TM \rightarrow M$ be the bundle projection. For some open neighbourhood $\mathcal{W} \subseteq \mathcal{D}$ of the zero-section, the map

$$A = (A_1, A_2): \mathcal{W} \rightarrow M \times M, \quad v \mapsto (\pi_{TM}(v), \exp(v)) \quad (86)$$

has open image and is a diffeomorphism onto its image; it is called a *local addition*. In particular, for each $p \in M$ the set $\mathcal{W}_p := \mathcal{V} \cap T_p M$ is open in $T_p M$ and the function $\exp_p := \exp|_{T_p M}$ restricts to a C^∞ -diffeomorphism from \mathcal{W}_p onto the open neighbourhood $\exp_p(\mathcal{W}_p)$ of p in M . Let us write $\mathcal{X}_c(M)$ for the space of compactly supported smooth vector fields on M . There is an open 0-neighbourhood $\mathcal{V} \subseteq \mathcal{X}_c(M)$ with $X(M) \subseteq \mathcal{W}$ for all $X \in \mathcal{V}$ such that

$$\mathcal{U} := \{\exp \circ X: X \in \mathcal{V}\} \subseteq \text{Diff}_c(M)$$

and the map

$$\Phi: \mathcal{U} \rightarrow \mathcal{V}, \quad \Phi(\phi)(p) := (\exp_p|_{\mathcal{W}_p})^{-1}(\phi(p))$$

is a chart for $\text{Diff}_c(M)$, with inverse

$$\Phi^{-1}: \mathcal{V} \rightarrow \mathcal{U}, \quad X \mapsto A_2 \circ X = \exp \circ X.$$

Since $\exp(0_p) = p$, we have $A_2(0_p) = p$ and $A^{-1}(p, p) = 0_p$, for each $p \in M$, entailing that

$$\Phi(\mathcal{U} \cap \text{Diff}_K(M)) = \mathcal{V} \cap \mathcal{X}_K(M).$$

Hence $\text{Diff}_K(M)$ is a submanifold of $\text{Diff}_c(M)$ modelled on the Fréchet space $\mathcal{X}_K(M)$ of smooth vector fields supported in K .

11.2 Let \mathcal{C} be the set of connected components C of M . The above set \mathcal{V} can be chosen as $\mathcal{V} = \bigoplus_{C \in \mathcal{C}} \mathcal{V}_C$ with open 0-neighbourhoods $\mathcal{V}_C \subseteq \mathcal{X}_c(C)$, making it clear that the weak direct product

$$\bigoplus_{C \in \mathcal{C}} \text{Diff}_c(C)$$

can be considered as an open subgroup of $\text{Diff}_c(M)$. By Proposition 8.2, the weak direct product (and hence also $\text{Diff}_c(M)$) will be L^1 -regular if we can show that $\text{Diff}_c(C)$ is L^1 -regular for each connected component $C \subseteq M$. Moreover, $\text{Diff}_K(M)$ has $\bigoplus_{C \in \mathcal{C}} \text{Diff}_{K \cap C}(C)$ as an open subgroup and hence is L^1 -regular if each $\text{Diff}_{K \cap C}(C)$ is so.

We may (and will) therefore assume throughout the rest of this section that M is a connected and σ -compact finite-dimensional smooth manifold.

Thus $\mathcal{X}_c(M)$ is a strict (LF)-space; if $(K_j)_{j \in \mathbb{N}}$ is a sequence of compact subsets $K_j \subseteq M$ such that $M = \bigcup_{j \in \mathbb{N}} K_j$ and $K_j \subseteq (K_{j+1})^0$ (the interior) for each $j \in \mathbb{N}$, then

$$\mathcal{X}_c(M) = \lim_{\rightarrow K} \mathcal{X}_K(M) = \lim_{\rightarrow j \in \mathbb{N}} \mathcal{X}_{K_j}(M).$$

11.3 We shall identify the Lie algebra of $G := \text{Diff}_c(M)$ with the locally convex space $\mathcal{X}_c(M)$ (with the negative of the traditional Lie bracket of vector fields) by means of the isomorphism

$$d\Phi|_{T_{\text{id}_M} G}: L(G) \rightarrow \mathcal{X}_c(M).$$

Likewise, we identify the Lie algebra of $G_K := \text{Diff}_K(M)$ with $\mathcal{X}_K(M)$. We shall see later that $\text{Diff}_c(M)$ is L^1 -regular, with smooth right evolution

$$\text{Evol}^r : L^1([0, 1], \mathcal{X}_c(M)) \rightarrow AC([0, 1], \text{Diff}_c(M)),$$

and observe in Remark 11.21 that $\text{Evol}^r([\lambda \circ \gamma])(t) \in \text{Diff}_K(M)$ for $[\gamma]$ in an open 0-neighbourhood Ω_K in $L^1([0, 1], \mathcal{X}_K(M))$, where $\lambda : \mathcal{X}_K(M) \rightarrow \mathcal{X}_c(M)$ is the inclusion map. Then

$$\text{Evol}^r([\gamma]) \in AC([0, 1], \text{Diff}_K(M)), \quad (87)$$

as the conclusion of Lemma 4.12 is available by Remark 4.14. Moreover, the map

$$h : \Omega_K \rightarrow AC([0, 1], \text{Diff}_K(M)), \quad [\gamma] \mapsto \text{Evol}^r([\gamma])$$

is smooth, as $AC([0, 1], \text{Diff}_K(M))$ is a submanifold of $AC([0, 1], \text{Diff}_c(M))$ since $\text{Diff}_K(M)$ is a submanifold of $\text{Diff}_c(M)$ (using that the conclusion of Lemma 4.13 applies by Remark 4.14). For $[\gamma] \in L^1([0, 1], \mathcal{X}_K(M))$, consider $\eta := \text{Evol}^r([\lambda \circ \gamma])$ as an element of $AC([0, 1], \text{Diff}_K(M))$, as in 87. If $j : G_K \rightarrow G$ is the inclusion map, then we can identify $L(j)$ with λ , and thus

$$L^1([0, 1], \lambda)(\delta_{G_K}^r(\eta)) = \delta_G^r(j \circ \eta) = [\lambda \circ \gamma] = L^1([0, 1], \lambda)([\gamma])$$

(where we wrote the Lie group G_K as an index for clarity). Hence $\delta_{G_K}^r(\eta) = [\gamma]$. We deduce that h is the right evolution map $\text{Evol}_{G_K}^r$ for $G_K = \text{Diff}_K(M)$ on Ω_K . Hence $\text{Diff}_K(M)$ is L^1 -regular, by Proposition 5.25. Thus, it only remains to show that $\text{Diff}_c(M)$ is L^1 -regular, and to show the validity of Remark 11.21.

The proof of the following proposition, which is similar to the familiar case of a C^k -curve γ (see, e.g., [22]; cf. [63] and [45]) has been relegated to the appendix (Appendix C).

Proposition 11.4 *Let $\gamma \in \mathcal{L}^1([0, 1], \mathcal{X}_c(M))$ such that*

$$f := \widehat{\gamma} : [0, 1] \times M \rightarrow TM, \quad (t, p) \mapsto \gamma(t)(p)$$

satisfies a local L^1 -Lipschitz condition. Let $\eta \in AC_{L^1}([0, 1], \text{Diff}_c(M))$ with $\eta(0) = \text{id}_M$. Then $\eta = \text{Evol}^r([\gamma])$ if and only if f admits global flow for initial time $t_0 = 0$ and

$$\eta(t)(p) = \Phi_{t,0}^f(p)$$

for all $t \in [0, 1]$ and $p \in M$, with notation as in Definition 10.6. □

Let n be the dimension of M .

11.5 Consider the continuous seminorm $p := \|\cdot\|_{\overline{B}_4(0), \|\cdot\|_{op}} \circ D + \|\cdot\|_{\overline{B}_4(0), \|\cdot\|_{\infty}}$ on $C^\infty(B_5(0), \mathbb{R}^n)$, where $D: C^\infty(B_5(0), \mathbb{R}^n) \rightarrow C^\infty(B_5(0), \mathbb{R}^{n \times n})$, $f \mapsto f'$; thus

$$p(f) = \sup_{x \in \overline{B}_4(0)} (\|f'(x)\|_{op} + \|f(x)\|_{\infty}) \quad \text{for } f \in C^\infty(B_5(0), \mathbb{R}^n).$$

Fix $L \in]0, 1[$. Then

$$Q := \{\gamma \in L^1([0, 1], C^\infty(B_5(0), \mathbb{R}^n)) : \|\gamma\|_{L^1, p} < L\}$$

is an open 0-neighbourhood in $L^1([0, 1], C^\infty(B_5(0), \mathbb{R}^n))$. We define a map $\Psi: Q \times B_3(0) \times C([0, 1], B_4(0)) \rightarrow C([0, 1], B_4(0))$ via

$$\Psi([\gamma], x, \kappa)(t) := x + \int_0^t \gamma(s)(\kappa(s)) ds$$

for $[\gamma] \in Q$ with $\gamma \in \mathcal{L}^1([0, 1], C^\infty(B_5(0), \mathbb{R}^n))$, $x \in B_3(0)$, $\kappa \in C([0, 1], B_4(0))$ and $t \in [0, 1]$ (recalling that the integrand always is an \mathcal{L}^1 -function of s by Lemma 9.13).

The following lemma can be proved exactly as Lemma 9.18 (increasing all radii by 2):

Lemma 11.6 *The map $\Psi: Q \times B_3(0) \times C([0, 1], B_4(0)) \rightarrow C([0, 1], B_4(0))$ is smooth and defines a uniform family of contractions in the final variable, in the sense that*

$$\text{Lip}(\Psi([\gamma], x, \bullet)) \leq L$$

for all $[\gamma] \in Q$ and all $x \in B_3(0)$. □

11.7 There exists a locally finite cover $(U_j)_{j \in J}$ of M by relatively compact, open subsets $U_j \subseteq M$ such that charts

$$\kappa_j: U_j \rightarrow B_5(0) \subseteq \mathbb{R}^n$$

with image $B_5(0)$ can be defined on U_j and the smaller sets $\kappa_j^{-1}(B_1(0))$ form an open cover of M (this follows from [47, Chapter II, Theorem 3.3], as there exist diffeomorphisms $B_3(0) \rightarrow B_5(0)$ which leave $B_1(0)$ invariant). Since we assume that M is σ -compact, the set J is countable.

11.8 For each $j \in J$ and vector field $X \in \mathcal{X}_c(M)$, we write

$$X^{(j)} := d\kappa_j \circ X \circ \kappa_j^{-1}: B_5(0) \rightarrow \mathbb{R}^n$$

for its representative in the local chart κ_j . Then

$$\mathcal{X}_c(M) \rightarrow C^\infty(B_5(0), \mathbb{R}^n), \quad X \mapsto X^{(j)}$$

is a continuous linear map. Moreover, for each $r \in [1, 5]$, the map

$$\rho_r: \mathcal{X}_c(M) \rightarrow \bigoplus_{j \in J} C^\infty(U_j, \mathbb{R}^n), \quad \rho(X) := (X^{(j)}|_{B_r(0)})_{j \in J}$$

is a linear topological embedding which admits a continuous linear right inverse, whence its image is closed and complemented as a topological vector space (see Lemma A.15). Hence also

$$\begin{aligned} R_5 &:= L^1([0, 1], \rho_5): L^1([0, 1], \mathcal{X}_c(M)) \rightarrow L^1\left([0, 1], \bigoplus_{j \in J} C^\infty(B_5(0), \mathbb{R}^n)\right) \\ &\cong \bigoplus_{j \in J} L^1([0, 1], C^\infty(B_5(0), \mathbb{R}^n)) \end{aligned}$$

and

$$\begin{aligned} R_1 &:= C([0, 1], \rho_1): C([0, 1], \mathcal{X}_c(M)) \rightarrow C\left([0, 1], \bigoplus_{j \in J} C^\infty(B_1(0), \mathbb{R}^n)\right) \\ &\cong \bigoplus_{j \in J} C([0, 1], C^\infty(B_1(0), \mathbb{R}^n)) \end{aligned}$$

are linear topological embeddings with closed (and complemented) image.

11.9 Let Q be as in 11.5; then

$$\mathcal{Q} := R_5^{-1}\left(\bigoplus_{j \in J} Q\right) = \{[\gamma] \in L^1([0, 1], \mathcal{X}_c(M)): (\forall j \in J) [t \mapsto (\gamma(t))^{(j)}] \in Q\}$$

is an open 0-neighbourhood in $L^1([0, 1], \mathcal{X}_c(M))$.

11.10 Let us write $R_5([\gamma]) = ([\gamma_j])_{j \in J}$ with $\gamma_j \in \mathcal{L}^1([0, 1], C^\infty(B_5(0), \mathbb{R}^n))$.

Lemma 11.11 *For each $[\gamma] \in \mathcal{Q}$, the map*

$$\widehat{\gamma}: [0, 1] \times M \rightarrow TM, \quad \widehat{\gamma}(t, p) := \gamma(t)(p)$$

satisfies a local L^1 -Lipschitz condition.

Proof. Abbreviate $f := \widehat{\gamma}$. If $p \in M$, then $p \in \kappa_j^{-1}(B_3(0))$ for some $j \in J$. Let $U_\kappa := \kappa_j^{-1}(B_3(0))$ and $\kappa := \kappa_j|_{U_\kappa}: U_\kappa \rightarrow B_2(0)$. Define

$$f_\kappa: [0, 1] \times B_3(0) \rightarrow \mathbb{R}^n, \quad f_\kappa(t, y) := d\kappa(f(t, \kappa^{-1}(y)))$$

as in Definition 10.3. For $t \in [0, 1]$, we have $f_\kappa(t, \bullet) = (\rho_j \circ \gamma)(t)|_{B_2(0)}$. Hence

$$\begin{aligned} \text{Lip}(f_\kappa(t, \bullet)) &= \text{Lip}(\rho_j \circ \gamma)(t)|_{B_3(0)} = \sup_{y \in B_3(0)} \|(\rho_j(\gamma(t)))'(y)\|_{op} \\ &\leq p(\rho_j(\gamma(t))) =: h(t) \end{aligned}$$

with

$$\int_0^1 h(t) dt = \|[\rho_j \circ \gamma]\|_{L^1, p} = \|L^1([0, 1], \rho_j)([\gamma])\|_{L^1, p} \leq L < 1$$

as $L^1([0, 1], \rho_j)([\gamma]) \in \mathcal{Q}$ by definition of \mathcal{Q} . □

11.12 As in 9.19, we see that if $([\gamma], x) \in \mathcal{Q} \times B_3(0)$, then the contraction $\Psi([\gamma], x, \bullet): C([0, 1], B_4(0)) \rightarrow C([0, 1], B_4(0))$ has a unique fixed point $\zeta_{[\gamma], x} \in C([0, 1], B_4(0))$. Thus

$$\Psi([\gamma], x, \zeta_{[\gamma], x}) = \zeta_{[\gamma], x}. \quad (88)$$

Since Ψ is smooth, Lemma 6.2 shows that also the map

$$\mathcal{Q} \times B_3(0) \rightarrow C([0, 1], B_4(0)), \quad ([\gamma], x) \mapsto \zeta_{[\gamma], x}$$

is smooth. Define $F([\gamma])(t)(x) := \zeta_{[\gamma], x}(t)$ for $[\gamma] \in \mathcal{Q}$, $x \in B_3(0)$ and $t \in [0, 1]$. Using the exponential laws from [2], we deduce:

- (a) $F([\gamma])(t) \in C^\infty(B_3(0), \mathbb{R}^n)$ for all $[\gamma] \in \mathcal{Q}$ and $t \in [0, 1]$;
- (b) $F([\gamma]) \in C([0, 1], C^\infty(B_3(0), \mathbb{R}^n))$ for all $[\gamma] \in \mathcal{Q}$;
- (c) $F: \mathcal{Q} \rightarrow C([0, 1], C^\infty(B_3(0), \mathbb{R}^n))$ is smooth.

Define $H([\gamma])(t) := F([\gamma])(t)|_{B_2(0)}$ for $[\gamma] \in Q$ and $t \in [0, 1]$. Thus

$$H = C([0, 1], \varrho) \circ F: Q \rightarrow C([0, 1], C^\infty(B_2(0), \mathbb{R}^n)),$$

where $\varrho: C^\infty(B_3(0), \mathbb{R}^n) \rightarrow C^\infty(B_2(0))$, $\kappa \mapsto \kappa|_{B_2(0)}$ and

$$C([0, 1], \varrho): C([0, 1], C^\infty(B_3(0), \mathbb{R}^n)) \rightarrow C([0, 1], C^\infty(B_2(0), \mathbb{R}^n)), \quad \kappa \mapsto \varrho \circ \kappa$$

are continuous linear maps (cf. [25]). Hence H is smooth, being a composition of smooth maps.

Lemma 11.13 *For each $[\gamma] \in Q$, the map $H([\gamma]): [0, 1] \rightarrow C^\infty(B_2(0), \mathbb{R}^n)$ is absolutely continuous.*

Proof. The set

$$Z := \{\kappa \in C^\infty(B_3(0), \mathbb{R}^n): \kappa(\overline{B}_2(0)) \subseteq B_4(0)\}$$

is an open 0-neighbourhood in $C^\infty(B_3(0), \mathbb{R}^n)$, whence $C([0, 1], Z)$ is an open 0-neighbourhood in $C([0, 1], C^\infty(B_3(0), \mathbb{R}^n))$. By [25, Lemma 11.4], the map

$$f: Z \times C^\infty(B_5(0), \mathbb{R}^n) \rightarrow C^\infty(B_2(0), \mathbb{R}^n), \quad f(\tau, \sigma) := \sigma \circ \tau$$

is smooth. Moreover, $f(\tau, \bullet)$ is linear for each $\tau \in Z$, and we have $F([\gamma]) \in Z$. Hence

$$f \circ (F([\gamma]), \gamma) \in \mathcal{L}^1([0, 1], C^\infty(B_2(0), \mathbb{R}^n)),$$

by Lemma 2.1 (b), entailing that $\kappa: [0, 1] \rightarrow C^\infty(B_2(0), \mathbb{R}^n)$,

$$\kappa(t) := \text{id}_{B_2(0)} + \int_0^t f(F([\gamma])(s), \gamma(s)) ds = \text{id}_{B_2(0)} + \int_0^t \gamma(s) \circ F([\gamma])(s) ds$$

is an absolutely continuous function. Hence, the proof will be complete if we can show that

$$H([\gamma]) = \kappa.$$

It suffices to show that $H([\gamma])(t)(x) = \kappa(t)(x)$ for each $t \in [0, 1]$ and $x \in B_2(0)$. Since $\varepsilon_x: C^\infty(B_2(0), \mathbb{R}^n) \rightarrow \mathbb{R}^n$, $\tau \mapsto \tau(x)$ is a continuous linear map, we have

$$\begin{aligned} \kappa(t)(x) &= \varepsilon_x(\kappa(t)) = x + \int_0^t \varepsilon_x(\gamma(s) \circ F([\gamma])(s)) ds \\ &= x + \int_0^t \gamma(s)(F([\gamma])(s)(x)) ds = x + \int_0^t \gamma(s)(\zeta_{[\gamma], x}) ds \\ &= \Psi([\gamma], x, \zeta_{[\gamma], x}(t)) = \zeta_{[\gamma], x}(t) = H([\gamma])(t)(x), \end{aligned}$$

exploiting for the second equality that weak integrals and continuous linear maps can be interchanged. This finishes the proof. \square

11.14 Given $[\gamma] \in \mathcal{Q}$, we have $[\gamma_j] \in \mathcal{Q}_j$ for each $j \in J$ (using the notation from 11.10). For $p \in M$, we have $p \in \kappa_j^{-1}(B_3(0))$ for some $j \in J$. Let $x := \kappa(p)$. By (88) and definition of Ψ , we have $\zeta_{[\gamma_j],x} \in AC_{L^1}([0, 1], B_4(0))$ and

$$\zeta_{[\gamma_j],x}(t) = x + \int_0^t \gamma_j(s)(\zeta_{[\gamma_j],x}(s)) ds \quad \text{for all } t \in [0, 1],$$

i.e., $\zeta_{[\gamma_j],x}$ is a Carathéodory solution to

$$y'(t) = \gamma_j(t)(y(t)) = \widehat{\gamma}_j(t, y(t)), \quad y(0) = x.$$

Hence

$$\eta_{[\gamma],p}: [0, 1] \rightarrow M, \quad t \mapsto \kappa_j^{-1}(\zeta_{[\gamma_j],x}(t)) \quad (89)$$

is a Carathéodory solution to

$$y'(t) = \gamma(t)(y(t)) = \widehat{\gamma}(t, y(t)), \quad y(0) = p;$$

since $\widehat{\gamma}$ satisfies a local L^1 -Lipschitz condition by the first half of Proposition 11.4, such solutions are unique by Proposition 10.5 and thus $\eta_{[\gamma],p}$ is well defined, independent of the choice of j . By the preceding, $\widehat{\gamma}$ admits a global flow for the initial time $t_0 = 0$, and the latter is given by

$$\Phi_{t,0}^{\widehat{\gamma}}(p) = \eta_{[\gamma],p}(t) \quad \text{for } t \in [0, 1], p \in M.$$

We shall also write

$$\eta(t)(p) \quad \text{or} \quad \eta_{[\gamma]}(t)(p) \quad (90)$$

for $\Phi_{t,0}^{\widehat{\gamma}}(p)$.

11.15 Since ρ_1 is a topological embedding, after shrinking \mathcal{V} we may assume that

$$\mathcal{V} = \rho_1^{-1} \left(\bigoplus_{j \in J} \mathcal{V}_j \right)$$

with suitable open zero-neighbourhoods $\mathcal{V}_j \subseteq C^\infty(B_1(0), \mathbb{R}^n)$.

11.16 For $j \in J$, we obtain a Riemannian metric g_j on $B_5(0) \subseteq \mathbb{R}^n$ via

$$g_j((x, y), (x, z)) := g(T\kappa_j^{-1}(x, y), T\kappa_j^{-1}(x, z))$$

for $x \in B_5(0)$, $y, z \in \mathbb{R}^n$. This metric makes κ_j an isometry. If $\exp_j: \mathcal{D}_j \rightarrow B_5(0)$ is the Riemannian exponential map for $(B_5(0), g_j)$, then

$$T\kappa_j^{-1}(\mathcal{D}_j) \subseteq \mathcal{D}$$

and

$$\kappa_j^{-1} \circ \exp_j = \exp \circ T\kappa_j^{-1}|_{\mathcal{D}_j}. \quad (91)$$

Let $\text{pr}_1: B_5(0) \times \mathbb{R}^n \rightarrow B_5(0)$ be the projection onto the first component. We consider the smooth map

$$(\text{pr}_1, \exp_j): \mathcal{D}_j \rightarrow B_5(0) \times B_5(0), \quad (x, y) \mapsto (x, \exp_j(x, y)).$$

For each $x \in \overline{B}_4(0) \times \{0\}$, the smooth map $(\text{pr}_1, \exp_j(x, y))$ has invertible derivative

$$(v, w) \mapsto (v, v + w)$$

at $(x, 0)$ and hence is a local diffeomorphism around $(x, 0)$. Since (pr_1, \exp_j) is injective on the compact set $\overline{B}_4(0) \times \{0\}$, there is an open subset $O_j \subseteq \mathcal{D}_j$ containing $\overline{B}_4(0) \times \{0\}$ such that $(\text{pr}_1, \exp_j)(O_j)$ is open in $B_5(0) \times B_5(0)$ and $\psi_j := (\text{pr}_1, \exp_j)|_{O_j}$ is a C^∞ -diffeomorphism onto its image (see, e.g., [15, Lemma 4.6]). After shrinking O_j if necessary, we may assume that

$$T\kappa_j^{-1}(O_j) \subseteq \mathcal{W} \quad (92)$$

(with \mathcal{W} as in (86)). As the compact set $\Delta := \{(x, x): x \in \overline{B}_4(0)\}$ is contained in the open set $(\text{pr}_1, \exp_j)(O_j)$, there is $s_j \in]0, 1]$ such that

$$\Delta + (\{0\} \times B_{s_j}(0)) = \bigcup_{x \in \overline{B}_3(0)} \{x\} \times B_{s_j}(x) \subseteq (\text{pr}_1, \exp_j)(O_j).$$

Thus ψ_j^{-1} restricts to a C^∞ -diffeomorphism of the form

$$(\text{id}_{B_4(0)}, \theta_j): \bigcup_{x \in B_4(0)} \{x\} \times B_{s_j}(x) \rightarrow \psi_j^{-1} \left(\bigcup_{x \in B_4(0)} \{x\} \times B_{s_j}(x) \right)$$

whose range is an open subset of O_j . As a consequence, $\exp(x, \bullet)$ takes the open 0-neighbourhood $O_{j,x} := \{y \in \mathbb{R}^n : (x, y) \in O_j\}$ diffeomorphically onto an open subset of $B_5(0)$ which contains $B_{s_j}(x)$, and

$$(\exp(x, \bullet)|_{O_{j,x}})^{-1}|_{B_{s_j}(x)} = \theta_j(x, \bullet). \quad (93)$$

11.17 Then also the map

$$h_j : B_2(0) \times B_{s_j}(0) \rightarrow B_4(0), \quad (x, z) \mapsto \theta_j(x, x + z)$$

is smooth, and

$$Z_j := \{\gamma \in C^\infty(B_2(0), \mathbb{R}^n) : \gamma(\overline{B}_1(0)) \subseteq B_{s_j}(0)\}$$

is an open subset of $C^\infty(B_2(0), \mathbb{R}^n)$. As a consequence, also

$$\text{id}_{B_2(0)} + Z_j = \{\gamma \in C^\infty(B_2(0), \mathbb{R}^n) : (\forall x \in \overline{B}_1(0)) \gamma(x) \in B_{s_j}(x)\}$$

is an open subset of $C^\infty(B_2(0), \mathbb{R}^n)$. Consider the map

$$(h_j)_* : Z_j \rightarrow C^\infty(B_1(0), \mathbb{R}^n)$$

determined by $(h_j)_*(\gamma)(x) := h_j(x, \gamma(x))$ for $\gamma \in Z_j$ and $x \in B_1(0)$. Then $(h_j)_*$ is smooth, by [25, Proposition 4.23]. Now consider the map

$$(\theta_j)_* : \text{id}_{B_2(0)} + Z_j \rightarrow C^\infty(B_1(0), \mathbb{R}^n)$$

determined by $(\theta_j)_*(\gamma)(x) := \theta_j(x, \gamma(x))$ for $\gamma \in \text{id}_{B_2(0)} + Z_j$ and $x \in B_1(0)$. Since $(\theta_j)_*(\gamma) = (h_j)_*(\gamma - \text{id}_{B_2(0)})$, also $(\theta_j)_*$ is smooth. In particular, $(\theta_j)_*$ is continuous. Since $(\theta_j)_*(\text{id}_{B_2(0)}) = 0$, there exists an open neighbourhood $Y_j \subseteq \text{id}_{B_2(0)} + Z_j$ of $\text{id}_{B_2(0)}$ such that

$$(\theta_j)_*(Y_j) \subseteq \mathcal{V}_j. \quad (94)$$

Since H is continuous by (11.12) and $H(0)(t) = \text{id}_{B_2(0)}$ for all $t \in [0, 1]$, there is an open 0-neighbourhood $P_j \subseteq Q$ such that

$$H(P_j) \subseteq C([0, 1], Y_j). \quad (95)$$

Now

$$\mathcal{P} := R_5^{-1} \left(\bigoplus_{j \in J} P_j \right)$$

is an open 0-neighbourhood in $L^1([0, 1], \mathcal{X}_c(M))$, and $\mathcal{P} \subseteq \mathcal{Q}$.

11.18 Given $[\gamma] \in \mathcal{P}$, write $R_5([\gamma]) = ([\gamma_j])_{j \in J}$ with $\gamma_j \in \mathcal{L}^1([0, 1], C^\infty(B_5(0), \mathbb{R}))$. Let $\eta(t) := \widehat{\Phi}_{t,0}^\gamma$. If $p \in M$ such that $p \in \kappa_j^{-1}(B_1(0))$, let $x := \kappa_j(p)$. Then

$$\begin{aligned} (\theta_j)_*(H([\gamma_j])(t))(x) &= \theta_j(x, H([\gamma_j])(t)(x)) = \theta_j(x, \zeta_{[\gamma_j],x}(t)) \\ &= \theta_j(\kappa_j(p), \kappa_j(\eta(t)(p))) = d\kappa_j(\exp_p|_{\mathcal{W}_p})^{-1}(\eta(t)(p)) \end{aligned}$$

by (91), (92), and (93). Therefore

$$(\theta_i)_*(H([\gamma_i])(t))(x) = d(\kappa_j \circ \kappa_i^{-1})((\theta_j)_*(H([\gamma_j])(t))(x))$$

if $p \in \kappa_j^{-1}(B_1(0)) \cap \kappa_i^{-1}(B_1(0))$. Thus

$$((\theta_j)_*(H([\gamma_j])(t)))_{j \in J} \in \text{im}(\rho_1)$$

and hence

$$((\theta_j)_*(H([\gamma_j])(t)))_{j \in J} \in \text{im}(R_1),$$

whence we find a unique $\theta = \theta_{[\gamma]} \in C([0, 1], \mathcal{X}_c(M))$ such that

$$R_1(\theta) = ((\theta_j)_*(H([\gamma_j])(t)))_{j \in J}.$$

Then $\rho_1(\theta(t)) = ((\theta_j)_*(H([\gamma_j])(t)))_{j \in J} \in \bigoplus_{j \in J} \mathcal{V}_j$, whence

$$\theta(t) \in \mathcal{V}$$

and hence

$$\exp \circ \theta(t) \in \text{Diff}_c(M).$$

Define $\eta(t) := \eta_{[\gamma]}(t) := \widehat{\Phi}_{t,0}^\gamma$. If $p \in M$ and $j \in J$ with $p \in \kappa_j^{-1}(B_1(0))$, then

$$\begin{aligned} \exp(\theta(t)(p)) &= \kappa_j^{-1} \exp_j T\kappa_j \theta(t)(\kappa_j^{-1}(x)) = \kappa_j^{-1} \exp_j (\theta_j)_*(H([\gamma_j])(t))(x) \\ &= \kappa_j^{-1} \exp_j (\exp_j|_{\mathcal{W}_j})^{-1} H([\gamma_j])(t)(x) = \kappa_j^{-1} H([\gamma_j])(t)(x) \\ &= \kappa_j^{-1}(\zeta_{[\gamma_j],x}(t)) = \widehat{\Phi}_{t,0}^\gamma(p) = \eta(t)(p) \end{aligned}$$

with $x := \kappa_j(p)$. Hence

$$\eta(t) = \exp \circ \theta(t) = \Phi(\theta(t)) \in \text{Diff}_c(M),$$

enabling us to consider η as a map $\eta: [0, 1] \rightarrow \text{Diff}_c(M)$.

11.19 Since $\eta = \Phi \circ \theta$, the map $\eta: [0, 1] \rightarrow \text{Diff}_c(M)$ will be absolutely continuous if we can show that $\theta: [0, 1] \rightarrow \mathcal{X}_c(M)$ is absolutely continuous, which is equivalent to absolute continuity of the map

$$[0, 1] \rightarrow \bigoplus_{j \in J} C^\infty(B_1(0), \mathbb{R}^n), \quad t \mapsto \rho_1(\theta(t)),$$

by Lemma 4.12 (since $\text{im}(\rho_1)$ is complemented in the direct sum). Note that $J_0 := \{j \in J: [\gamma_j] \neq 0\}$ is a finite set. If $j \in J \setminus J_0$, then $\zeta_{[\gamma_j], x}(t) = x$ for all $x \in B_2(0)$ and $t \in [0, 1]$, entailing that

$$\eta(t)(p) = p$$

for all $p \in$ (see (89) and (90)) Therefore

$$\theta(t)(p) = (\exp_p|_{\mathcal{W}_p})^{-1}(\eta(t)(p)) = 0$$

for all $j \in J \setminus J_0$, $t \in [0, 1]$, and $p \in U_j$. Hence

$$\text{im}(\rho_1 \circ \theta) \subseteq \bigoplus_{j \in J_0} C^\infty(B_1(0), \mathbb{R}^n)$$

and thus $\rho_1 \circ \theta$ (and hence θ) will be absolutely continuous if we can show that its components

$$[0, 1] \rightarrow C^\infty(B_1(0), \mathbb{R}^n), \quad t \mapsto (\theta_j)_*(H([\gamma_j])(t))$$

are absolutely continuous for all $j \in J_0$. Since $(\theta_j)_*$ is smooth, it suffices to know that $H([\gamma_j])$ is absolutely continuous for each $j \in J_0$. But this is the case by Lemma 11.13. Hence η is absolutely continuous, and thus

$$\eta = \text{Evol}^r([\gamma]),$$

by Proposition 11.4.

11.20 In view of Proposition 5.25 and the final assertion of Proposition 5.20, to complete the proof of L^1 -regularity of $\text{Diff}_c(M)$, it only remains to check that the map

$$\mathcal{P} \rightarrow C([0, 1], \text{Diff}_c(M)), \quad [\gamma] \mapsto \text{Evol}^r([\gamma])$$

is smooth. Since $\text{Evol}^r([\gamma]) = \eta_{[\gamma]} = \Phi \circ \theta_{[\gamma]} = C([0, 1], \Phi)(\eta_{[\gamma]})$, where $C([0, 1], \Psi: C([0, 1], \mathcal{V}) \rightarrow C([0, 1], \mathcal{U})$ is smooth (being a the inverse of a chart for the Lie group $C([0, 1], \text{Diff}_c(M))$), we need only show that

$$\mathcal{P} \rightarrow C([0, 1], \mathcal{X}_c(M)), \quad [\gamma] \mapsto \theta_{[\gamma]}$$

is smooth. This will hold if the map

$$\mathcal{P} \rightarrow \bigoplus_{j \in J} C([0, 1], C([0, 1], C^\infty(B_1(0), \mathbb{R}^n))), \quad [\gamma] \mapsto R_1(\theta_{[\gamma]}) \quad (96)$$

is smooth. But

$$R_1(\theta_{[\gamma]}) = ((\theta_j)_*(H([\gamma_j])))_{j \in J} = (\oplus_{j \in J} ((\theta_j)_* \circ H)) (R_5([\gamma]),$$

showing that the map in (96) coincides with

$$(\oplus_{j \in J} ((\theta_j)_* \circ H)) \circ R_5|_{\mathcal{P}}.$$

This map is smooth since R_5 is continuous linear and $\oplus_{j \in J} ((\theta_j)_* \circ H)$ is smooth (by [24, Proposition 7.1]) as each of the maps $(\theta_j)_* \circ H$ is so.

Remark 11.21 Given a compact subset $K \subseteq M$, let

$$[\gamma] \in \Omega_K := \mathcal{L}^1([0, 1], \mathcal{X}_K(M)) \cap \mathcal{P}.$$

If $p \in K$, choose $j \in J$ with $p \in \kappa_j^{-1}(B_1(0))$ and set $x := \kappa_j(p)$. Then

$$\begin{aligned} \text{Evol}^r([\gamma])(t)(p) &= \eta_{[\gamma]}(t)(p) = \eta_{[\gamma], t}(p) \\ &= \kappa_j^{-1}(\zeta_{[\gamma], x}(t)) - \kappa_j^{-1}(x) = p \end{aligned}$$

for each $t \in [0, 1]$, using that $\zeta_{[\gamma], x}(t) = x$ for all t since also the constant function $t \mapsto x$ is a fixed point of

$$\Psi([\gamma_j], x, \bullet) : \kappa \mapsto x + \int_0^t \gamma_j(s)(\kappa(s)(x)) ds$$

(as $\gamma(s)(x) = 0$). Hence $\text{Evol}^r([\gamma]) \in C([0, 1], \text{Diff}_K(M))$, as used in 11.3.

12 Consequences of regulated regularity

In this section, we prove Theorems H and I from the introduction, devoted to the strong Trotter property and the strong commutator property (as defined in the introduction). We begin with a lemma from [38] due to K.-H. Neeb.

Lemma 12.1 *Let E be a locally convex space, $U \subseteq E$ be an open set, $r > 0$, $\gamma: [0, r] \rightarrow U$ be a C^1 -curve and $f: U \rightarrow F$ be a C^2 -map with $df(\gamma(0), \cdot) = 0$. Then*

$$\eta: [0, r^2] \rightarrow U, \quad t \mapsto f(\gamma(\sqrt{t}))$$

is C^1 with $\eta'(0) = \frac{1}{2}d^2f(\gamma(0), \gamma'(0), \gamma'(0))$.

Proof. We may assume that $\gamma(0) = 0$ and $f(0) = 0$. Noting that

$$\gamma(\sqrt{t}) = \sqrt{t} \frac{\gamma(\sqrt{t}) - \overbrace{\gamma(\sqrt{0})}^{=0}}{\sqrt{t}} = \sqrt{t} \gamma^{[1]}(0, 1, \sqrt{t}),$$

we get for $t \in]0, r^2]$

$$\begin{aligned} \eta'(t) &= \frac{1}{2\sqrt{t}} df(\gamma(\sqrt{t}); \gamma'(\sqrt{t})) - \underbrace{\frac{1}{2\sqrt{t}} df(0, \gamma'(\sqrt{t}))}_{=0} \\ &= \frac{1}{2} (df)^{[1]}(0, \gamma'(\sqrt{t}); \gamma^{[1]}(0, 1, \sqrt{t}), 0; \sqrt{t}) \end{aligned}$$

The right-hand-side makes sense also for $t = 0$ and is continuous on $[0, r^2]$. So η is C^1 and $\eta'(0) = \frac{1}{2}(df)^{[1]}(0, \gamma'(0); \gamma'(0), 0; 0) = \frac{1}{2}d^2f(0, \gamma'(0), \gamma'(0))$. \square

The following consequence (see also [54, proof of Proposition II.6.3]) is relevant for our ends.

Lemma 12.2 *If G is a Lie group, $r > 0$ and $\gamma_1, \gamma_2 \in C^1([0, r], G)$ with $\gamma_1(0) = \gamma_2(0) = e$, then $\eta: [0, r^2] \rightarrow G$,*

$$\eta(t) := \gamma_1(\sqrt{t})\gamma_2(\sqrt{t})\gamma_1(\sqrt{t})^{-1}\gamma_2(\sqrt{t})^{-1}$$

is C^1 , and $\eta'(0) = [\gamma'_1(0), \gamma'_2(0)]$ in the Lie algebra $L(G) = T_e(G)$.

Proof. It is clear that $\eta|_{[0,r^2]}$ is C^1 . Let $U \subseteq G$, $V \subseteq U$ be open identity neighbourhoods with $VV^{-1}V^{-1} \subseteq U$. Identify U with an open set in E using a chart, such that $e = 0$. The map

$$f: V \times V \rightarrow U, \quad f(x, y) := xyx^{-1}y^{-1}$$

is smooth with $df(0, 0, v, w) = 0$ and

$$d^2f(0, 0; x, y; x, y) = 2[x, y]. \quad (97)$$

After shrinking r , we may assume that $\eta([0, r^2]) \subseteq U$. The assertions now follow from Lemma 12.1 and (97). \square

Proof of Theorem H. Assume that G has the strong Trotter property and let $\gamma, \eta: [0, 1] \rightarrow G$ be C^1 -curves such that $\gamma(0) = \eta(0) = e$. Then

$$\zeta: [0, 1] \rightarrow G, \quad \zeta(t) := \gamma(\sqrt{t})\eta(\sqrt{t})(\gamma(\sqrt{t}))^{-1}(\eta(\sqrt{t}))^{-1}$$

is a C^1 -curve with $\zeta'(0) = [\gamma'(0), \eta'(0)]$ (see Lemma 12.2). By the strong Trotter property,

$$\begin{aligned} & \left(\gamma\left(\frac{\sqrt{t}}{n}\right)\eta\left(\frac{\sqrt{t}}{n}\right)\gamma\left(\frac{\sqrt{t}}{n}\right)^{-1}\eta\left(\frac{\sqrt{t}}{n}\right)^{-1} \right)^{n^2} \\ &= \zeta_{n^2}(t/n^2) \rightarrow \exp_G(t\zeta'(0)) = \exp_G(t[\gamma'(0), \eta'(0)]) \end{aligned}$$

as $n \rightarrow \infty$, uniformly in compact subsets of $[0, \infty[$. \square

Proof of Theorem I. If $\zeta: [0, 1] \rightarrow G$ is a C^1 -curve with $\zeta(0) = e$ and $m \in \mathbb{N}$, we define $\zeta_n: [0, m] \rightarrow G$ via

$$\zeta_n(t) := (\zeta(t/n))^n$$

for $n \in \mathbb{N}$ such that $n \geq m$. We claim that

$$\zeta_n(t) \rightarrow \exp_G(t\zeta'(0)) \quad \text{as } n \rightarrow \infty, \quad (98)$$

uniformly in $t \in [0, m]$ (entailing that G has the strong Trotter property).

To establish the claim, let U be an open identity neighbourhood in G . We show that there exists $n_0 \geq m$ such that

$$(\forall n \geq n_0)(\forall t \in [0, m]) \quad \zeta_n(t) \in U \exp_G(t\zeta'(0)) \cap \exp_G(t\zeta'(0))U. \quad (99)$$

For $v \in \mathfrak{g}$, let $c_v: [0, 1] \rightarrow \mathfrak{g}$ be the constant curve given by

$$c_v(s) := v \quad \text{for all } s \in [0, 1].$$

Because the map $\mathfrak{g} \rightarrow C([0, 1], \mathfrak{g}) \subseteq \mathcal{R}([0, 1], \mathfrak{g})$, $v \mapsto c_v$ is continuous, the set

$$K := \{c_{t\zeta'(0)} : t \in [0, m]\}$$

is compact. Also the map $\text{evol}: \mathcal{R}([0, 1], \mathfrak{g}) \rightarrow G$, $\text{evol}(\sigma) := \text{Evol}(\sigma)(1)$ is continuous. Hence

$$\text{evol}(K) \subseteq G$$

is compact, whence there exists an open identity neighbourhood $V \subseteq U$ such that $gVg^{-1} \subseteq U$ and thus $gV \subseteq gU \cap Ug$, for all $g \in \text{evol}(K)$. Since $\text{evol}(c_{t\zeta'(0)}) = \exp_G(t\zeta'(0))$, we deduce that

$$\exp_G(t\zeta'(0))V \subseteq U \exp_G(t\zeta'(0)) \cap \exp_G(t\zeta'(0))U, \quad \text{for all } t \in [0, m]. \quad (100)$$

Next, we show that there is an open 0-neighbourhood Q in $\mathcal{R}([0, 1], \mathfrak{g})$ such that

$$\text{evol}(\theta + \sigma) \in \text{evol}(\theta)V \quad \text{for all } \theta \in K \text{ and } \sigma \in Q. \quad (101)$$

To this end, let $W \subseteq G$ be an open identity neighbourhood such that $W^{-1}W \subseteq V$. Again using that $\text{evol}: \mathcal{R}([0, 1], \mathfrak{g}) \rightarrow G$ is continuous, for each $\theta \in K$ we find an open 0-neighbourhood $P_\theta \subseteq \mathcal{R}([0, 1], \mathfrak{g})$ such that

$$\text{evol}(\theta + P_\theta) \subseteq \text{evol}(\theta)W.$$

Let $Q_\theta \subseteq \mathcal{R}([0, 1], \mathfrak{g})$ be an open 0-neighbourhood such that $Q_\theta + Q_\theta \subseteq P_\theta$. Then $K \subseteq \bigcup_{j=1}^\ell (\theta_j + Q_{\theta_j})$ for some finite subset $\{\theta_1, \dots, \theta_\ell\} \subseteq K$. Moreover,

$$Q := \bigcap_{j=1}^\ell Q_{\theta_j}$$

is an open 0-neighbourhood in $\mathcal{R}([0, 1], \mathfrak{g})$. Then (101) holds. In fact, for $\theta \in K$ we have $\theta \in \theta_j + Q_j$ for some $j \in \{1, \dots, \ell\}$. Since $\theta - \theta_j \in Q_j \subseteq P_j$, we have

$$\text{evol}(\theta) = \text{evol}(\theta_j + (\theta - \theta_j)) \in \text{evol}(\theta_j)W$$

and thus

$$\text{evol}(\theta_j) \in \text{evol}(\theta)W^{-1}. \quad (102)$$

For $\sigma \in Q \subseteq Q_j$, we have $(\theta - \theta_j) + \sigma \in Q_j + Q_j \subseteq P_j$ and thus $\text{evol}(\theta + \sigma) = \text{evol}(\theta_j + (\theta - \theta_j) + \sigma) \in \text{evol}(\theta_j)W \subseteq \text{evol}(\theta)W^{-1}W \subseteq \text{evol}(\theta)V$, using (102) for the penultimate inclusion. Thus (101) is established.

For $t \in [0, m]$, consider the continuous curve $\alpha_{n,t}: [0, 1] \rightarrow G$ defined piecewise for $s \in [k/n, (k+1)/n]$ with $k \in \{0, \dots, n-1\}$ as

$$\alpha_{n,t}(s) := \zeta(t/n)^k \zeta((s - k/n)t). \quad (103)$$

Then $\alpha_{n,t}|_{[k/n, (k+1)/n]}$ is C^1 for each $k \in \{1, \dots, n-1\}$, entailing that $\alpha_{n,t} \in AC_{\mathcal{R}}([0, 1], G)$ with $\beta_{n,t} := \delta^\ell(\alpha_{n,t}) \in \mathcal{R}([0, 1], \mathfrak{g})$. By construction,

$$\zeta_n(t) = \alpha_{n,t}(1) = \text{evol}(\beta_{n,t}).$$

The explicit formula (103) for $\alpha_{n,t}|_{[k/n, (k+1)/n]}$ shows that

$$\beta_{n,t}(s) = t\delta^\ell(\zeta)((s - k/n)t) \quad (104)$$

for all $k \in \{0, 1, \dots, n-1\}$ and $t \in [k/n, (k+1)/n]$. We have

$$\{\tau \in \mathcal{R}([0, 1], \mathfrak{g}) : \|\tau\|_{L^\infty, q} \leq 1\} \subseteq Q$$

for a continuous seminorm q on \mathfrak{g} . Since $\delta^\ell \zeta: [0, 1] \rightarrow \mathfrak{g}$ is continuous, there is $\varepsilon \in]0, 1]$ such that

$$q(\delta^\ell \zeta(x) - \delta^\ell(\zeta(0))) \leq \frac{1}{m} \quad \text{for all } x \in [0, \varepsilon]. \quad (105)$$

Choose $n_0 \geq m$ so large that $\frac{m}{n_0} \leq \varepsilon$. Let $n \geq n_0$. Then

$$(s - \frac{k}{n})t \leq \frac{m}{n_0} \leq \varepsilon \quad (106)$$

for all $t \in [0, m]$ and $s \in [0, 1]$, with $k \in \{0, 1, \dots, n-1\}$ such that $s \in [k/n, (k+1)/n]$. Combining (106) with (105) and (104), we see that

$$\begin{aligned} \beta_{n,t}(s) &= t\delta^\ell(\zeta)(0) + t(\delta^\ell(\zeta)((s - k/n)t) - \delta^\ell(\zeta)(0)) \\ &\in t\delta^\ell(\zeta)(0) + t\overline{B}_{1/m}^q(0) \subseteq t\delta^\ell(\zeta)(0) + \overline{B}_1^q(0). \end{aligned}$$

Since $\delta^\ell(\zeta)(0) = \zeta'(0)$, we deduce that

$$\|\beta_{n,t} - c_{t\zeta'(0)}\|_{L^\infty, q} \leq 1$$

for all $n \geq n_0$ and $t \in [0, m]$. Thus $\beta_{n,t} - c_{t\zeta'(0)} \in Q$ and hence

$$\begin{aligned} \zeta_n(t) &= \text{evol}(\beta_{n,t}) = \text{evol}(c_{t\zeta'(0)} + (\beta_{n,t} - c_{t\zeta'(0)})) \\ &\in \text{evol}(c_{t\zeta'(0)})V = \exp_G(t\zeta'(0))V \subseteq U \exp_G(t\zeta'(0)) \cap \exp_G(t\zeta'(0))U \end{aligned}$$

for all $n \geq n_0$ and $t \in [0, m]$, using (101) and (100). Thus (99) holds and the proof is complete. \square

A Proofs for Section 1

We provide proofs for results compiled in the indicated section, and some auxiliary results.

Proof of 1.2. (a) The topology on X is initial with respect to the mappings ϕ_j for $j \in J$. Therefore, intersections

$$\bigcap_{j \in F} \phi_j^{-1}(U_j)$$

form a basis of open neighbourhoods of x , for finite subsets $F \subseteq J$ and open neighbourhoods $U_j \subseteq X_j$ of $\phi_j(x)$ for $j \in F$. Since J is directed, we find $j_0 \in J$ such that $j \leq j_0$ for all $j \in F$. Then $U := \bigcap_{j \in F} (\phi_{j,j_0})^{-1}(U_j)$ is an open neighbourhood of $\phi_{j_0}(x)$ in X_{j_0} and $(\phi_{j_0})^{-1} \subseteq \bigcap_{j \in F} \phi_j^{-1}(U_j)$, as $\phi_j(\phi_{j_0}^{-1}(U)) = \phi_{j,j_0}(\phi_{j_0}(\phi_{j_0}^{-1}(U))) \subseteq \phi_{j,j_0}(U) \subseteq \phi_{j,j_0}(\phi_{j,j_0}^{-1}(U_j)) \subseteq U_j$ for each $j \in F$.

(b) If D is dense in X , then $\phi_j(D)$ is dense in $\phi_j(X)$, by continuity. Conversely, assume that $\phi_j(D)$ is dense in $\phi_j(X)$ for all $j \in J$. Let $x \in X$ and V be an open neighbourhood of x in X . By (a), we may assume that $V = \phi_j^{-1}(U)$ for some $j \in J$ and open neighbourhood U of $\phi_j(x)$ in X_j . Since $\phi_j(D)$ is dense in $\phi_j(X)$, we find $y \in D$ such that $\phi_j(y) \in U$. Then $y \in \phi_j^{-1}(U) = V$ and thus D is dense in X . \square

Lemma A.1 *Let (X, Σ) be a measurable space, $n \in \mathbb{N}$, Y_1, \dots, Y_n be metric spaces and $\gamma_j: (X, \Sigma) \rightarrow (Y_j, \mathcal{B}(Y_j))$ be a measurable map with separable image, for $j \in \{1, \dots, n\}$. Then the following holds:*

- (a) $\gamma := (\gamma_1, \dots, \gamma_n): X \rightarrow Y_1 \times \dots \times Y_n$ is measurable with respect to the σ -algebra $\mathcal{B}(Y)$ on the direct product topological space $Y := Y_1 \times \dots \times Y_n$.
- (b) If Z is a topological space and $f: Y_1 \times \dots \times Y_n \rightarrow Z$ a continuous map, then the map $f \circ (\gamma_1, \dots, \gamma_n): (X, \Sigma) \rightarrow (Z, \mathcal{B}(Z))$ is measurable.

Proof. (a) By 1.6 (b), we need only show that γ is measurable as a map to $\text{im}(\gamma_1) \times \dots \times \text{im}(\gamma_n)$, equipped with the trace of $\mathcal{B}(Y)$. The latter coincides with the Borel- σ -algebra with respect to the induced topology \mathcal{O} (see 1.6 (d)). But \mathcal{O} coincides with the product topology on $\gamma_1(X) \times \dots \times \gamma_n(X)$, if we use the topology induced by Y_j on $\gamma_j(X)$, for each j . After replacing Y_j with $\gamma_j(X)$ if necessary, we may therefore assume that each Y_j is separable

and hence second countable. As a consequence, $\mathcal{B}(Y_1 \times \cdots \times Y_n) = \mathcal{B}(Y_1) \otimes \cdots \otimes \mathcal{B}(Y_n)$ is the product σ -algebra (see 1.6 (f)). But γ is measurable with respect to this product σ -algebra, as all of its components γ_j are measurable (see 1.6 (e)).

(b) The maps $\gamma: (X, \Sigma) \rightarrow (Y, \mathcal{B}(Y))$ and $f: (Y, \mathcal{B}(Y)) \rightarrow (Z, \mathcal{B}(Z))$ are measurable, hence also $f \circ \gamma$. \square

Remark A.2 The preceding lemma entails that $\mathcal{L}^p(X, \mu, E)$ and $\mathcal{L}_{rc}^\infty(X, \mu, E)$ are vector subspaces of E^X , for each Fréchet space E and $p \in [1, \infty]$.

The next two lemmas and Lemma A.4 (a) will help us to prove Lemma 1.19. Afterwards, we develop machinery for the proof of Lemma 1.28.

Lemma A.3 *Let (X, Σ) be a measurable space, (Y, d) be a metric space and $\gamma_n: X \rightarrow Y$ be a measurable function with separable image, for each $n \in \mathbb{N}$.*

- (a) *If $\gamma(x) := \lim_{n \rightarrow \infty} \gamma_n(x)$ exists for each $x \in X$, then $\gamma: X \rightarrow Y$, $x \mapsto \gamma(x)$ is a measurable map with separable image.*
- (b) *If the metric space (Y, d) is complete, then*

$$C := \{x \in X : \gamma_n(x) \text{ converges as } n \rightarrow \infty\}$$

is a measurable set.

Proof. After replacing Y with the closure of $\bigcup_{n \in \mathbb{N}} \gamma_n(X)$, we may assume that the metric space Y is separable (see 1.6 (b) and (d)).

(a) Is a special case of [24, Lemma 2.5].

(b) Since Y is separable and hence second countable, we have $\mathcal{B}(Y \times Y) = \mathcal{B}(Y) \otimes \mathcal{B}(Y)$ (see, e.g., [24, Lemma 2.7]). For $\ell \in \mathbb{N}$, the set

$$O_\ell := \{(y, z) \in Y \times Y : d(y, z) < 1/\ell\}$$

is open in $Y \times Y$. Hence $O_\ell \in \mathcal{B}(Y \times Y) = \mathcal{B}(Y) \otimes \mathcal{B}(Y)$. Fix $x \in X$. Because Y is complete, the sequence $(\gamma_n(x))_{n \in \mathbb{N}}$ converges if and only if it is a Cauchy sequence, requiring that for each $\ell \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that $(\gamma_n(x), \gamma_m(x)) \in O_\ell$ for all $n, m \geq N$. Hence

$$C = \bigcap_{\ell \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{n, m \geq N} (\gamma_n, \gamma_m)^{-1}(O_\ell).$$

Since $(\gamma_n, \gamma_n): (X, \Sigma) \rightarrow (Y \times Y, \mathcal{B}(Y) \otimes \mathcal{B}(Y))$ is a measurable map (as it has measurable components) and each O_ℓ is measurable, C is measurable. \square

Proof of Lemma 1.11. For $m \in \mathbb{N}$, let K_m be the m -fold sum

$$K_m := [-m, m]K + \cdots + [-m, m]K := \{x_1 + \cdots + x_m : x_1, \dots, x_m \in [-m, m]K\}.$$

Lemma 1.10 implies that K_m is compact and metrizable, as it is the image of a continuous map $[-m, m]^m \times K^m \rightarrow E$ on the metrizable compact space $[-m, m]^m \times K^m$. Let d_m be a metric on K_m defining its topology and set

$$K_{m,j} := \{(x, y) \in K_m \times K_m : d_m(x, y) \geq 1/j\}$$

for $j \in \mathbb{N}$. Then $\bigcup_{j \in \mathbb{N}} K_{m,j} = (K_m \times K_m) \setminus \Delta_m$, where $\Delta_m := \{(x, x) : x \in K_m\} \subseteq K_m \times K_m$ is the diagonal. Moreover, each of the sets $K_{m,j}$ is compact. For each $(x, y) \in K_{m,j}$, we have $x \neq y$ and hence find a continuous seminorm $q_{m,j,x,y} \in P(E)$ such that $q_{m,j,x,y}(x - y) \neq 0$ and thus $q_{m,j,x,y}(x - y) = 1$ without loss of generality. Then the sets

$$B_{1/2}^{q_{m,j,x,y}}(x) \times B_{1/2}^{q_{m,j,x,y}}(y)$$

form an open cover of $K_{m,j}$ for $(x, y) \in K_{m,j}$, and hence there is a finite subset $F_{m,j} \subseteq K_{m,j}$ such that

$$K_{m,j} \subseteq \bigcup_{(x,y) \in F_{m,j}} (B_{1/2}^{q_{m,j,x,y}}(x) \times B_{1/2}^{q_{m,j,x,y}}(y)).$$

Thus, if $(v, w) \in K_{m,j}$, we find $(x, y) \in F_{m,j}$ such that

$$(v, w) \in B_{1/2}^{q_{m,j,x,y}}(x) \times B_{1/2}^{q_{m,j,x,y}}(y) =: V \times W. \quad (107)$$

Then V and W are open neighbourhoods of v and w , respectively, in E . Since $q(y - x) = 1$, these neighbourhoods are disjoint (otherwise a contradiction would result from the triangle inequality). Now $\Gamma := \{q_{m,j,x,y} : m, j \in \mathbb{N}, (x, y) \in F_{m,j}\} \subseteq P(E)$ is a countable set,

$$E_K := \text{span}(K) = \bigcup_{m \in \mathbb{N}} K_m \quad \text{and} \quad \bigcup_{m,j \in \mathbb{N}} K_{m,j} = (E_K \times E_K) \setminus \Delta,$$

where $\Delta := \{(x, x) : x \in E_K\}$ is the diagonal in $E_K \times E_K$. We give E_K the locally convex vector topology \mathcal{O}' defined by the countable set Γ of seminorms. Then \mathcal{O}' is Hausdorff, since any $(v, w) \in (E_K \times E_K) \setminus \Delta$ is contained

in $K_{m,j}$ for some $m, j \in \mathbb{N}$ and hence $E_K \cap B_{1/2}^{q_{m,j,x,y}}(x)$ and $E_K \cap B_{1/2}^{q_{m,j,x,y}}(y)$ (with notation as in (107)) are disjoint open neighbourhoods of v and w in (E_K, \mathcal{O}) . \square

Let $\mathcal{F}(X, \Sigma, E)$ be the space of all measurable maps $\gamma: (X, \Sigma) \rightarrow (E, \mathcal{B}(E))$ with finite image. Then $\mathcal{F}(X, \Sigma, E) \cap \mathcal{L}^1(X, \mu, E)$ is the vector subspace of all $\gamma \in \mathcal{F}(X, \Sigma, E)$ such that $\mu(\gamma^{-1}(E \setminus \{0\})) < \infty$.

If X is a locally compact topological space, we write $C_c(X, E)$ for the space of all compactly supported continuous E -valued functions on X . Recall that a *Radon measure on X* is a measure $\mu: \mathcal{B}(X) \rightarrow [0, \infty[$ which is finite on compact sets and inner regular (see, e.g. [4]).

Lemma A.4 *Let (X, Σ, μ) be a measure space E be a Fréchet space. Then:*

- (a) $\mathcal{F}(X, \Sigma, E)$ is dense in $\mathcal{L}_{rc}^\infty(X, \Sigma, E)$ and $\mathcal{F}(X, \Sigma, E) \cap \mathcal{L}^1(X, \mu, E)$ is dense in $\mathcal{L}^1(X, \mu, E)$.
- (b) The weak integral

$$\int_X \gamma(x) d\mu(x)$$

exists in E , for each $\gamma \in \mathcal{L}^1(X, \mu, E)$, and

$$p\left(\int_X \gamma(x) d\mu(x)\right) \leq \int_X p(\gamma(x)) d\mu(x) \quad (108)$$

for each continuous seminorm p on E .

- (c) If X is a locally compact space and μ a Radon measure on X , then $C_c(X, E)$ is dense in $\mathcal{L}^1(X, \mu, E)$.

Proof. (a) Let p be a continuous seminorm on E , and $\varepsilon > 0$. If $\gamma \in \mathcal{L}^1(X, \mu, E)$, then $X_n := \{x \in X: p(\gamma(x)) \geq \frac{1}{n}\} \in \Sigma$ (cf. Remark A.2) and $(p \circ \gamma)^{-1}([0, \infty]) = \bigcup_{n \in \mathbb{N}} X_n$, where $X_1 \subseteq X_2 \subseteq \dots$. Thus $\mu(X_n) < \infty$, as

$$\frac{1}{n} \mu(X_n) \leq \int_{X_n} p(\gamma(x)) d\mu(x) \leq \int_X p(\gamma(x)) d\mu(x) < \infty.$$

Moreover,

$$\begin{aligned} \|\gamma\|_{\mathcal{L}^1, p} &= \int_X p(\gamma(x)) d\mu(x) = \int_{(p \circ \gamma)^{-1}([0, \infty])} p(\gamma(x)) d\mu(x) \\ &= \lim_{n \rightarrow \infty} \int_{X_n} p(\gamma(x)) d\mu(x), \end{aligned}$$

whence there exists $N \in \mathbb{N}$ such that

$$\int_{X \setminus X_N} p(\gamma(x)) d\mu(x) < \frac{\varepsilon}{3}$$

for all $n \geq N$. If $\mu(X_N) = 0$, then $\eta := 0$ is an element of $\mathcal{F}(X, \Sigma, E) \cap \mathcal{L}^1(X, \mu, E)$ such that $\|\gamma - \eta\|_{\mathcal{L}^1, p} < \varepsilon$. If $m := \mu(X_N) > 0$, let $D := \{y_n : n \in \mathbb{N}\}$ be a countable dense subset of $\gamma(X)$. Then

$$\gamma(X_N) \subseteq \overline{\gamma(X)} = \overline{D} \subseteq D + B_{\varepsilon/(3m)}^p(0) = \bigcup_{n \in \mathbb{N}} B_{\varepsilon/(3m)}^p(y_n).$$

We define $A_1 := \{x \in X_N : \gamma(x) \in B_{\varepsilon/(3m)}^p(y_1)\} \in \Sigma$ and $A_n := \{x \in X_N : \gamma(x) \in B_{\varepsilon/(3m)}^p(y_n)\} \setminus \bigcup_{k=1}^{n-1} A_k \in \Sigma$ for integers $n \geq 2$. Then

$$X_N = \bigcup_{n \in \mathbb{N}} A_n$$

is a countable union of disjoint sets. Arguing as above, we find $n_0 \in \mathbb{N}$ such that

$$\int_{X_N \setminus \bigcup_{n=1}^{n_0} A_n} p(\gamma(x)) d\mu(x) < \frac{\varepsilon}{3}.$$

Define $\eta := \sum_{n=1}^{n_0} y_n \mathbf{1}_{A_n}$, using the characteristic function $\mathbf{1}_{A_n} : X \rightarrow \{0, 1\}$ of A_n . Then $\eta \in \mathcal{F}(X, \Sigma, E) \cap \mathcal{L}^1(X, \mu, E)$ and

$$\begin{aligned} \|\gamma - \eta\|_{\mathcal{L}^1, p} &= \int_X p(\gamma(x) - \eta(x)) d\mu(x) \\ &= \int_{X \setminus X_N} p(\gamma(x)) d\mu(x) + \int_{X_N \setminus \bigcup_{n=1}^{n_0} A_n} p(\gamma(x)) d\mu(x) \\ &\quad + \int_{\bigcup_{n=1}^{n_0} A_n} \underbrace{p(\gamma(x) - \eta(x))}_{\leq \varepsilon/(3m)} d\mu(x) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3m} \underbrace{\mu\left(\bigcup_{n=1}^{n_0} A_n\right)}_{\leq \mu(X_N)=m} < \varepsilon. \end{aligned}$$

Hence $\mathcal{F}(X, \Sigma, E) \cap \mathcal{L}^1(X, \mu, E)$ is dense in $\mathcal{L}^1(X, \mu, E)$.

If $\gamma \in \mathcal{L}_{rc}^\infty(X, \mu, E)$, then $\gamma(X)$ is precompact in E , whence there exists a finite set $\{y_1, \dots, y_n\} \subseteq \gamma(X)$ such that

$$\gamma(x) \subseteq \bigcup_{k=1}^n B_\varepsilon^p(y_k).$$

Then $A_1 := \gamma^{-1}(B_\varepsilon^p(y_1)) \in \Sigma$ and $A_k := \gamma^{-1}(B_\varepsilon^p(y_k)) \setminus (A_1 \cup \dots \cup A_{k-1}) \in \Sigma$ for $k \in \{2, \dots, n\}$. Moreover,

$$\eta := \bigcup_{k=1}^n y_k \mathbf{1}_{A_k} \in \mathcal{F}(X, \Sigma, E)$$

and $\|\gamma - \eta\|_{\mathcal{L}^{\infty, p}} < \varepsilon$ by construction. Hence $\mathcal{F}(X, \Sigma, E)$ is dense in $\mathcal{L}_{rc}^\infty(X, \mu, E)$.

(b) If $\gamma \in \mathcal{F}(X, \Sigma, E) \cap \mathcal{L}^1(X, \mu, E)$, then $\gamma = \sum_{k=1}^n y_k \mathbf{1}_{A_k}$ for some $n \in \mathbb{N}_0$, $y_k \in E$ and disjoint sets $A_1, \dots, A_n \in \Sigma$ such that $\mu(A_k) < \infty$ for each $k \in \{1, \dots, n\}$ (for example, if $\gamma(X) = \{y_1, \dots, y_n\}$ with pairwise distinct elements y_1, \dots, y_n , we can take $A_k := \gamma^{-1}(\{y_k\})$). We define

$$I(\gamma) := \sum_{k=1}^n \mu(A_k) y_k$$

(declaring the empty sum as 0, if $n = 0$). Without changing (γ) , we may omit those indices k such that $y_k = 0$ or $A_k = \emptyset$. We may therefore assume that $y_k \neq 0$ and $A_k \neq \emptyset$ for each $k \in \{1, \dots, n\}$. Then $I(\gamma)$ is well-defined. In fact, assume that also $\gamma = \sum_{j=1}^m z_j \mathbf{1}_{B_j}$ with disjoint measurable subsets B_1, \dots, B_m of finite measure, such that each z_j is non-zero and each $B_j \neq \emptyset$. Then

$$\bigcup_{k=1}^n A_k = \{x \in X : \gamma(x) \neq 0\} = \bigcup_{j=1}^m B_j. \quad (109)$$

If $k \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, then either $A_k \cap B_j = \emptyset$ (whence $\mu(A_k \cap B_j) = 0$ and thus $y_k \mu(A_k \cap B_j) = 0 = z_j \mu(A_k \cap B_j)$) or there exists $c \in A_k \cap B_j$, whence $y_k = \gamma(c) = z_j$ and again $y_k \mu(A_k \cap B_j) = z_j \mu(A_k \cap B_j)$. Hence, using that $A_k = \bigcup_{j=1}^m A_k \cap B_j$ (by (109) which is a disjoint union,

$$\sum_{k=1}^n y_k \mu(A_k) = \sum_{k=1}^n \sum_{j=1}^m y_k \mu(A_k \cap B_j) = \sum_{k=1}^n \sum_{j=1}^m z_j \mu(A_k \cap B_j) = \sum_{j=1}^m z_j \mu(B_j).$$

If also $\eta \in \mathcal{F}(X, \Sigma, E) \cap \mathcal{L}^1(X, \mu, E)$ and $r, s \in \mathbb{R}$, write $\eta = \sum_{i=1}^{\ell} w_i \mathbf{1}_{C_i}$ with disjoint measurable subsets C_i of finite measure and $w_i \in E$. Then $\gamma = \sum_{k=1}^n \sum_{i=1}^{\ell} y_k \mathbf{1}_{A_k \cap C_i}$. Writing η and $r\gamma + s\eta$ likewise as a linear combination of the characteristic functions $\mathbf{1}_{A_k \cap C_i}$, we easily find that

$$I(r\gamma + s\eta) = rI(\gamma) + sI(\eta).$$

Thus I is linear. Moreover, I is continuous with respect to the topology induced by $\mathcal{L}^1(X, \mu, E)$ on $\mathcal{F}(X, \Sigma, E) \cap \mathcal{L}^1(X, \mu, E)$, as

$$p(I(\gamma)) = p\left(\sum_{k=1}^n \mu(A_k) y_k\right) \leq \sum_{k=1}^n p(y_k) \mu(A_k) = \int_X p(\gamma(x)) d\mu(x) = \|\gamma\|_{\mathcal{L}^1, p}$$

for each γ as above and each continuous seminorm p on E . Since E is complete and $\mathcal{F}(X, \Sigma, E) \cap \mathcal{L}^1(X, \mu, E)$ is dense in $\mathcal{L}^1(X, \mu, E)$, the continuous linear map I has a unique continuous linear extension

$$J: \mathcal{L}^1(X, \mu, E) \rightarrow E.$$

For each continuous linear functional $\lambda: E \rightarrow \mathbb{R}$, both $\lambda \circ J$ and the map

$$h: \mathcal{L}^1(X, \mu, E) \rightarrow \mathbb{R}, \quad \gamma \mapsto \int_X \lambda(\gamma(x)) d\mu(x)$$

are continuous linear extensions of $\lambda \circ I$, whence $J = h$ by density of $\mathcal{F}(X, \Sigma, E) \cap \mathcal{L}^1(X, \mu, E)$. Thus

$$\lambda(J(\gamma)) = \int_X \lambda(\gamma(x)) d\mu(x)$$

for each $\lambda \in E'$ and thus $J(\gamma)$ is the weak integral $\int_X \gamma(x) d\mu(x)$ in E . If p is a continuous seminorm on E , then $\mathcal{L}^1(X, \mu, E) \rightarrow \mathbb{R}$, $\gamma \mapsto p(J(\gamma))$ and $\gamma \mapsto \|\gamma\|_{\mathcal{L}^1, p}$ are continuous functions on $\mathcal{L}^1(X, \mu, E)$ such that $p(J(\gamma)) \leq \|\gamma\|_{\mathcal{L}^1, p}$ for each γ in the dense subset $\mathcal{F}(X, \Sigma, E) \cap \mathcal{L}^1(X, \mu, E)$. Hence $p(J(\gamma)) \leq \|\gamma\|_{\mathcal{L}^1, p}$ for all $\gamma \in \mathcal{L}^1(X, \mu, E)$, establishing (108).

(c) Since $\mathcal{F}(X, \mathcal{B}(X), E) \cap \mathcal{L}^1(X, \mu, E)$ is dense in $\mathcal{L}^1(X, \mu, E)$ and every γ in the former space is a linear combination of maps of the form $v \mathbf{1}_A$ with $A \in \mathcal{B}(X)$ and $\mu(A) < \infty$, it suffices to show that $v \mathbf{1}_A$ is in the closure of $C_c(X, E)$. Since μ is inner regular, there exists an ascending sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets of A such that $\mu(K_n) \geq \mu(A) - \frac{1}{n}$ and

thus $\mu(A \setminus K_n) \rightarrow 0$ as $n \rightarrow \infty$. Let $V_n \subseteq X$ be a relatively compact, open subset of X such that $K_n \subseteq V_n$. Since $\mu|_{\overline{V_n}}$ is outer regular, there exists a relatively open subset $W_n \subseteq \overline{V_n}$ such that $K_n \subseteq W_n$ and $\mu(W_n) \leq \mu(K_n) + \frac{1}{n}$. After replacing W_n with its intersection with V_n , we may assume that W_n is open in C . Now Urysohn's Lemma [60, 2.12] provides $\eta_n \in C_c(X, \mathbb{R})$ such that $\mathbf{1}_{K_n} \leq \eta_n \leq \mathbf{1}_{W_n}$. Then $v\eta_n \in C_c(X, E)$. If q is a continuous seminorm on E , then

$$\begin{aligned} \|v\eta_n - v\mathbf{1}_{K_n}\|_{\mathcal{L}^1, p} &= q(v) \int_{V_n \setminus K_n} |\eta_n(x)| d\mu(x) \\ &\leq q(v)(\mu(V_n \setminus A_n) + \mu(A_n \setminus K_n)) \leq \frac{2q(v)}{n}, \end{aligned}$$

which tends to 0 as $n \rightarrow \infty$. \square

Proof of Lemma 1.19. By Lemma A.4, the weak integral $\int_X \gamma d\mu$ exists in E for each $\gamma \in \mathcal{L}^1(X, \mu, E)$, and can be estimated as desired. The remaining assertions on $L_{rc}^\infty(X, E)$ are covered by [24, Proposition 3.21].³³ If E is a Banach space (resp., a Fréchet space), then $L^p(X, \mu, E)$ are normable (resp., have the property that the vector topology can be defined using a countable set of seminorms). It only remains to show that $L^p(X, \mu, E)$ is complete. To this end, let $(q_n)_{n \in \mathbb{N}}$ be a sequence of seminorms on E defining its vector topology. Let $\gamma_n \in \mathcal{L}^p(X, \mu, E)$ such that $([\gamma_n])_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^p(X, \mu, E)$. The Cauchy sequence will converge if we can show that it has a convergent subsequence. After passing to a subsequence, we may therefore assume that

$$(\forall n_0 \in \mathbb{N}) (\forall n, m \geq n_0) \quad \|\gamma_n - \gamma_m\|_{\mathcal{L}^p, q_{n_0}} \leq 2^{-n_0}.$$

We claim that $\sum_{n=1}^\infty (\gamma_{n+1} - \gamma_n)$ converges in $L^p(X, \mu, E)$. If this is true, then $\gamma_m = \gamma_1 + \sum_{n=1}^{m-1} (\gamma_{n+1} - \gamma_n)$ converges to $\gamma_1 + \sum_{n=1}^\infty (\gamma_{n+1} - \gamma_n)$ as $m \rightarrow \infty$, whence $L^p(X, \mu, E)$ is complete. We first prove the claim if $p = \infty$. After replacing the representatives by 0 on a set of measure 0, we may assume that $\|\gamma_n - \gamma_m\|_{\mathcal{L}^\infty, q_{n_0}} = \sup_{x \in X} q_{n_0}(\gamma_n(x) - \gamma_m(x))$ for all $n, m \geq n_0$. Let $x \in X$. For each $n_0 \in \mathbb{N}$, we have

$$\sum_{n=n_0}^\infty q_{n_0}(\gamma_{n+1}(x) - \gamma_n(x)) \leq \sum_{n=n_0}^\infty 2^{-n} < \infty$$

³³Where the symbol L^∞ is used in place of L_{rc}^∞ .

and thus $\sum_{n=1}^{\infty} q_{n_0}(\gamma_{n+1}(x) - \gamma_n(x)) < \infty$. Therefore $\sum_{n=1}^{\infty} (\gamma_{n+1}(x) - \gamma_n(x))$ is an absolutely convergent series in the Fréchet space E and hence convergent. By Lemma A.3, the function

$$\gamma: X \rightarrow E, \quad \gamma(x) := \sum_{n=1}^{\infty} (\gamma_{n+1}(x) - \gamma_n(x)) = \lim_{N \rightarrow \infty} \sum_{n=1}^N (\gamma_{n+1}(x) - \gamma_n(x))$$

is measurable and has separable image. For each $n_0 \in \mathbb{N}$ and $x \in X$, we have

$$\begin{aligned} q_{n_0}(\gamma(x)) &\leq \sum_{n=1}^{\infty} q_{n_0}(\gamma_{n+1}(x) - \gamma_n(x)) \\ &= \sum_{n=1}^{n_0-1} q_{n_0}(\gamma_{n+1}(x) - \gamma_n(x)) + \sum_{n=n_0}^{\infty} q_{n_0}(\gamma_{n+1}(x) - \gamma_n(x)) \\ &\leq \sum_{n=1}^{n_0-1} \|\gamma_n\|_{\mathcal{L}^{\infty, q_{n_0}}} + \sum_{n=n_0}^{\infty} 2^{-n}. \end{aligned}$$

As a consequence, $\text{im}(\gamma)$ is bounded in E and thus $\gamma \in \mathcal{L}^{\infty}(X, \mu, E)$, with $\|\gamma\|_{\mathcal{L}^{\infty, q_{n_0}}} \leq \sum_{n=1}^{n_0-1} \|\gamma_n\|_{\mathcal{L}^{\infty, q_{n_0}}} + \sum_{n=n_0}^{\infty} 2^{-n}$. If $n_0 \in \mathbb{N}$, we have for all $m \in \mathbb{N}$ with $m \geq n_0 - 2$ and $x \in X$

$$\begin{aligned} q_{n_0}(\gamma(x) - \sum_{n=0}^m (\gamma_{n+1}(x) - \gamma_n(x))) &\leq \sum_{n=m+2}^{\infty} q_{n_0}(\gamma_{n+1}(x) - \gamma_n(x)) \\ &\leq \sum_{n=m+2}^{\infty} 2^{-n} = \frac{2^{-m-2}}{1 - 1/2} = 2^{-m-1}. \end{aligned}$$

Thus $\|\gamma - \sum_{n=1}^m (\gamma_{n+1} - \gamma_n)\|_{\mathcal{L}^{\infty, q_{n_0}}} \leq 2^{-m-1}$ tends to 0 as $m \rightarrow \infty$ and hence $\gamma = \lim_{m \rightarrow \infty} \sum_{n=1}^m (\gamma_{n+1} - \gamma_n)$ in $\mathcal{L}^{\infty}(X, \mu, E)$ (establishing the claim).

If $p \in [1, \infty[$, for each $n_0 \in \mathbb{N}$ and $N \in \mathbb{N}$ we have

$$\begin{aligned} &\sqrt[p]{\int_X \left(\sum_{n=1}^N q_{n_0}(\gamma_{n+1} - \gamma_n) \right)^p d\mu} \\ &= \left\| \sum_{n=1}^N q_{n_0} \circ (\gamma_{n+1} - \gamma_n) \right\|_{\mathcal{L}^p} \leq \sum_{n=1}^N \|q_{n_0} \circ (\gamma_{n+1} - \gamma_n)\|_{\mathcal{L}^p} \\ &= \sum_{n=1}^N \|\gamma_{n+1} - \gamma_n\|_{\mathcal{L}^p, q_{n_0}} \leq \sum_{n=1}^{\infty} \|\gamma_{n+1} - \gamma_n\|_{\mathcal{L}^p, q_{n_0}}. \end{aligned}$$

Letting $N \rightarrow \infty$, the Monotone Convergence Theorem entails that

$$\sqrt[p]{\int_X \left(\sum_{n=1}^{\infty} q_{n_0}(\gamma_{n+1} - \gamma_n) \right)^p d\mu} \leq \sum_{n=1}^{\infty} \|\gamma_{n+1} - \gamma_n\|_{\mathcal{L}^p, q_{n_0}} < \infty.$$

Hence $\left(\sum_{n=1}^{\infty} q_{n_0}(\gamma_{n+1} - \gamma_n) \right)^p \in \mathcal{L}^1(X, \mathbb{R})$. Hence, after replacing each of the maps γ_n by 0 on a set of measure zero, we may assume that

$$\sum_{n=1}^{\infty} q_{n_0}(\gamma_{n+1} - (x)\gamma_n(x)) < \infty$$

for all $n_0 \in \mathbb{N}$ and all $x \in X$. Hence, for each $x \in X$, the series $\sum_{n=1}^{\infty} (\gamma_{n+1}(x) - \gamma_n(x))$ in E is absolutely convergent and hence convergent to some $\gamma(x) \in E$. By Lemma A.3, the function

$$\gamma: X \rightarrow E, \quad \gamma(x) := \sum_{n=1}^{\infty} (\gamma_{n+1}(x) - \gamma_n(x)) = \lim_{N \rightarrow \infty} \sum_{n=1}^N (\gamma_{n+1}(x) - \gamma_n(x))$$

is measurable and has separable image. Since

$$\begin{aligned} \int_X (q_{n_0}(\gamma(x)))^p d\mu(x) &\leq \int_X \left(\sum_{n=1}^{\infty} q_{n_0}(\gamma_{n+1}(x) - \gamma_n(x)) \right)^p d\mu(x) \\ &\leq \left(\sum_{n=1}^{\infty} \|\gamma_{n+1} - \gamma_n\|_{\mathcal{L}^p, q_{n_0}} \right)^p < \infty \end{aligned}$$

for each $n_0 \in \mathbb{N}$, we have $\gamma \in \mathcal{L}^p(X, \mu, E)$. Finally, $\sum_{n=1}^m (\gamma_{n+1} - \gamma_n) \rightarrow \gamma$ in $\mathcal{L}^p(X, \mu, E)$ as $m \rightarrow \infty$ since

$$\left\| \gamma - \sum_{n=1}^m (\gamma_{n+1} - \gamma_n) \right\|_{\mathcal{L}^p, q_{n_0}} = \sqrt[p]{\int_X q_{n_0} \left(\sum_{n=n_0}^{\infty} (\gamma_{n+1}(x) - \gamma_n(x)) \right)^p d\mu} \rightarrow 0$$

by dominated convergence, using that the integrands are majorized by the integrable function $\left(\sum_{n=1}^{\infty} q_{n_0} \circ (\gamma_{n+1} - \gamma_n) \right)^p$. \square

Proof of Lemma 1.21. Let $M_+^1(K)$ be the set of all Radon probability measures on K . Endow $M_+^1(K)$ with the vague topology, which makes it a

compact topological space and turns the map $\Phi: M_+^1(K) \rightarrow C(K)', \mu \mapsto I_\mu$ (with $I_\mu(f) := \int_K f d\mu$) into a topological embedding with respect to the weak $*$ -topology on the dual $C(K)'$ of the Banach space $C(K)$ of continuous real-valued functions on K with the supremum norm (see [4, Chapter 2, Corollary 4.7]). Because K is metrizable, $M_+^1(K)$ is metrizable (see [3, Satz 31.5 (a)]). The weak $*$ -topology on $C(K)'$ is initial with respect to the linear maps

$$\varepsilon_\gamma: C(K)' \rightarrow \mathbb{R}, \quad \lambda \mapsto \lambda(\gamma).$$

Hence $\varepsilon_\gamma \circ \Phi: M_+^1(K) \rightarrow \mathbb{R}, \mu \mapsto \int_K \gamma d\mu$ is continuous for each $\gamma \in C(K)$. For each $\mu \in M_+^1(K)$, the barycentre

$$b_\mu := \int_K x d\mu(x)$$

exists in the compact set $\overline{\text{conv}(K)}$ (see [4, Chapter 2, Proposition 5.3] or [61, Theorem 3.27]). The map

$$\beta: M_+^1(K) \rightarrow \overline{\text{conv}(K)}, \quad \mu \mapsto b_\mu$$

is continuous: Because $\overline{\text{conv}(K)} =: L$ is compact, the topology induced by E on L coincides with the weak topology, which is initial with respect to the mappings $\lambda|_L$ for $\lambda \in E'$. Hence β will be continuous if we can show that $\lambda \circ \beta$ is continuous for each $\lambda \in E'$. But

$$(\lambda \circ \beta)(\mu) = \int_K \lambda(x) d\mu(x) = I_\mu(\lambda|_L),$$

showing that $\lambda \circ \beta = \varepsilon_{\lambda|_L} \circ \Phi$, which indeed is continuous. Now Lemma 1.10 shows that $\text{im}(\beta)$ is compact and metrizable. It only remains to recall that $\text{im}(\beta) = \overline{\text{conv}(K)}$, see [4, Chapter 2, Proposition 5.3]. \square

Proof of Lemma 1.23. If $\mu = 0$, then all assertions are trivial. We may therefore assume that $\mu(X) > 0$. As weak integrals are linear in μ , we may assume that $\mu(X) = 1$. If $\gamma \in \mathcal{L}_{rc}^\infty(X, \mu, E)$, then $K := \text{im}(\gamma) \subseteq E$ compact and metrizable. Since E is assumed integral complete, it satisfies the metric CCP. Thus $\overline{\text{conv}(K)}$ is compact. Now consider the image measure $\gamma_*(\mu): \mathcal{B}(\overline{\text{conv}(K)}) \rightarrow [0, 1]$ of μ under the measurable map $\gamma: X \rightarrow \overline{\text{conv}(K)}$; thus $\gamma_*(\mu)(A) := \mu(\gamma^{-1}(A))$ for Borel sets $A \subseteq \overline{\text{conv}(K)}$. By [61, Theorem 3.27], the weak integral

$$b := \int_{\overline{\text{conv}(K)}} x d\gamma_*(\mu)(x)$$

exists in E . Testing with continuous linear functions $\lambda \in E'$ and using the Transformation Theorem for integrals with respect to image measures ([3, 19.2, Korollar 1]), we see that $b = \int_X \gamma d\mu$. If $q \in P(E)$, the Hahn-Banach theorem provides $\lambda \in E'$ such that $q(b) = \lambda(b)$ and $\lambda(B_1^q(0)) \subseteq [-1, 1]$. Thus $q(b) = \lambda(b) = \int_X \lambda(\gamma(x)) d\mu \leq \int_X |\lambda(\gamma(x))| d\mu \leq \int_X q(\gamma(x)) d\mu = \|\gamma\|_{\mathcal{L}^1, q} \leq \|\gamma\|_{\mathcal{L}^\infty, q}(X)$. \square

Let $\|\cdot\| := \|\cdot\|_2$ be the euclidean norm on \mathbb{R}^k and let $B_r(x) := \{y \in \mathbb{R}^k : \|y - x\| < r\}$ for $x \in \mathbb{R}^k$ and $r > 0$. Abbreviate $B_r := B_r(0)$. Then $\lambda_k(B_r(x)) = \lambda_k(B_r)$ for all $x \in \mathbb{R}^k$. The following discussion of Lebesgue points and absolutely continuous functions was inspired by the treatment of the scalar-valued case in [60, §7].

Definition A.5 Let E be a Fréchet space and $\gamma \in \mathcal{L}^1(\mathbb{R}^k, \lambda_k, E)$. A point $x \in \mathbb{R}^k$ is called a *Lebesgue point* of γ if

$$\lim_{r \rightarrow 0} \frac{1}{\lambda_k(B_r)} \int_{B_r(x)} p(\gamma(y) - \gamma(x)) d\lambda_k(y) = 0$$

for each continuous seminorm p on E .

Remark A.6 If $x \in \mathbb{R}^k$ is a Lebesgue point for $\gamma \in \mathcal{L}^1(\mathbb{R}^k, \lambda_k, E)$, then

$$\lim_{r \rightarrow 0} \frac{1}{\lambda_k(B_r)} \int_{B_r(x)} \gamma(y) d\lambda_k(y) = \gamma(x)$$

in particular, as

$$\begin{aligned} & p \left(\frac{1}{\lambda_k(B_r)} \int_{B_r(x)} \gamma(y) d\lambda_k(y) - \gamma(x) \right) \\ &= p \left(\frac{1}{\lambda_k(B_r)} \int_{B_r(x)} (\gamma(y) - \gamma(x)) d\lambda_k(y) \right) \\ &\leq \frac{1}{\lambda_k(B_r)} \int_{B_r(x)} p(\gamma(y) - \gamma(x)) d\lambda_k(y) \rightarrow 0 \end{aligned}$$

as $r \rightarrow 0$, for each continuous seminorm p on E .

Lemma A.10 implies that the same Lebesgue points are obtained if the euclidean norm is replaced with any norm $\|\cdot\|$ on \mathbb{R}^k .

Definition A.7 Let $\rho: \mathbb{R}^k \rightarrow [0, \infty]$ be a measurable function. We write $M_\rho: \mathbb{R}^k \rightarrow [0, \infty]$ for the *maximal function* of the measure $\rho d\lambda_k$, defined via

$$M_\rho(x) := \sup_{r \in]0, \infty[} \frac{1}{\lambda_k(B_r)} \int_{B_r(x)} \rho(y) d\lambda_k(y).$$

Then M_ρ is lower semicontinuous and hence Borel measurable (see [60, 7.2]).

Lemma A.8 *If E is a Fréchet space and $\gamma \in \mathcal{L}^1(\mathbb{R}^k, \lambda_k, E)$, then the set L_γ of all Lebesgue points of γ is a Borel set in \mathbb{R}^k , and $\lambda_k(\mathbb{R}^k \setminus L_\gamma) = 0$.*

Proof. Let $q_1 \leq q_2 \leq \dots$ be an ascending sequence of continuous seminorms on E defining the locally convex vector topology of E . Let $j \in \mathbb{N}$. For $r > 0$, the map $h_r: E \times \mathbb{R}^k \rightarrow \mathcal{L}^1(\mathbb{R}^k, E)$, $h_r(v, x) := v \mathbf{1}_{B_r(x)}$ is continuous, and also the map

$$\mathbb{R}^k \rightarrow \mathcal{L}^1(\mathbb{R}^k, E), \quad x \mapsto \gamma \cdot \mathbf{1}_{B_r(x)}$$

is continuous (exploiting that $\lambda_k(B_r(y) \Delta B_r(x)) \rightarrow 0$ as $r \rightarrow 0$, where $A \Delta B := (A \cup B) \setminus (A \cap B)$ denotes the symmetric difference). Hence

$$\mathbb{R}^k \rightarrow \mathcal{L}^1(\mathbb{R}^k, E), \quad x \mapsto \gamma(x) \mathbf{1}_{B_r(x)} - \gamma \cdot \mathbf{1}_{B_r(x)} = h_r(\gamma(x), x) - \mathbf{1}_{B_r(x)}$$

is a measurable map. Since $\|\cdot\|_{\mathcal{L}^1, q_j}$ is a continuous seminorm, we deduce that the map

$$g_{j,r}: \mathbb{R}^k \rightarrow \mathbb{R}, \quad x \mapsto \|\gamma(x) \mathbf{1}_{B_r(x)} - \gamma \cdot \mathbf{1}_{B_r(x)}\|_{\mathcal{L}^1, q_j} = \int_{B_r(x)} q_j(\gamma(x) - \gamma(y)) d\lambda_k(y)$$

is measurable. Hence also $f_{j,r}: \mathbb{R}^k \rightarrow \mathbb{R}$,

$$f_{j,r}(x) := \frac{1}{\lambda_k(B_r)} \int_{B_r(x)} q_j(\gamma(x) - \gamma(y)) d\lambda_k(y)$$

is measurable. We shall later write $f_{j,r,\gamma} := f_{j,r}$ to emphasize the dependence on γ . For fixed $x \in \mathbb{R}^k$, the map

$$]0, \infty[\rightarrow [0, \infty[, \quad r \mapsto g_{j,r}(x) = \int_{\mathbb{R}^k} q_j(\gamma(x) \mathbf{1}_{B_r(x)} - \gamma(y) \mathbf{1}_{B_r(y)}) d\lambda_k(y)$$

is continuous, exploiting that $\lambda_k(B_r(x))$ is continuous in r . Hence also the map

$$]0, \infty[\rightarrow [0, \infty[, \quad r \mapsto f_{j,r}(x) = \frac{1}{\lambda_k(B_r)} \int_{\mathbb{R}^k} q_j(\gamma(x) \mathbf{1}_{B_r(x)} - \gamma(y) \mathbf{1}_{B_r(y)}) d\lambda_k(y)$$

is continuous and thus

$$T_{j,r}(x) := \sup\{f_{j,s}(x) : s \in]0, r]\} = \sup\{f_{j,s}(x) : s \in]0, r] \cap \mathbb{Q}\}.$$

Being the pointwise supremum of a countable family of measurable functions, the function

$$T_{j,r} :]0, \infty[\rightarrow [0, \infty]$$

is measurable. By definition, $T_t \leq T_r$ if $t \leq r$. Hence

$$\begin{aligned} L_\gamma &= \{x \in \mathbb{R}^k : (\forall j \in \mathbb{N}) T_{j,r}(x) \rightarrow 0 \text{ as } r \rightarrow 0\} \\ &= \bigcap_{j \in \mathbb{N}} \{x \in \mathbb{R}^k : T_{j,1/n}(x) \rightarrow 0 \text{ as } n \rightarrow \infty\}. \end{aligned}$$

As the set A_j of all $x \in \mathbb{R}^k$ such that $(T_{j,1/n}(x))_{n \in \mathbb{N}}$ converges is a Borel set (see Lemma A.3 (b)) and $T_{j,\gamma} : A_j \rightarrow [0, \infty]$, $x \mapsto \lim_{n \rightarrow \infty} T_{j,1/n}(x)$ is a measurable map, we see that $L_\gamma = \bigcap_{j \in \mathbb{N}} T_{j,\gamma}^{-1}(\{0\})$ is a Borel set in \mathbb{R}^k .

Let $\varepsilon > 0$. Let $j, n \in \mathbb{N}$. By Lemma A.4 (c), there exists $\eta \in C_c(\mathbb{R}^k, E)$ such that $\|\gamma - \eta\|_{\mathcal{L}^1, q_j} < \frac{1}{n}$. Put $\xi := \gamma - \eta$. Since η is continuous, $T_{j,\eta} = 0$. Because

$$f_{j,r,\zeta}(x) \leq \frac{1}{\lambda_k(B_r)} \int_{B_r(x)} q_j(\zeta(y)) d\lambda_k(y) + p_j(\zeta(x)),$$

we have $T_{j,\zeta} \leq M_{q_j \circ \zeta} + p_j \circ \zeta$. Since $f_{j,r,\gamma} \leq f_{j,r,\eta} + f_{j,r,\xi}$, it follows that

$$T_{j,\gamma} \leq M_{p_j \circ \xi} + p_j \circ \xi.$$

Therefore

$$\begin{aligned} &\{x \in \mathbb{R}^k : T_{j,\gamma}(x) > 2\varepsilon\} \\ &\subseteq \{x \in \mathbb{R}^k : M_{p_j \circ \xi}(x) > \varepsilon\} \cup \{x \in \mathbb{R}^k : p_j(\xi(x)) > \varepsilon\} =: S_{\varepsilon,n}. \end{aligned}$$

Since $\|p_j \circ \xi\|_{\mathcal{L}^1} = \|\xi\|_{\mathcal{L}^1, p_j} < \frac{1}{n}$, [60, 7.5 (1) and Theorem 7.4] show that

$$\lambda_k(S_{\varepsilon,n}) \leq \frac{3^k}{\varepsilon} \|p_j \circ \xi\|_{\mathcal{L}^1} + \frac{1}{\varepsilon} \|p_j \circ \xi\|_{\mathcal{L}^1} \leq \frac{3^k + 1}{\varepsilon n}.$$

Hence $\lambda_k(\{x \in \mathbb{R}^k : T_{j,\gamma}(x) > 2\varepsilon\}) \leq \frac{3^k + 1}{\varepsilon n}$. As n was arbitrary, $\lambda_k(\{x \in \mathbb{R}^k : T_{j,\gamma}(x) > 2\varepsilon\}) = 0$ follows, and we deduce that

$$\lambda_k(\{x \in \mathbb{R}^k : T_{j,\gamma}(x) > 0\}) = 0.$$

Thus also $\mathbb{R}^k \setminus L_\gamma = \bigcup_{j \in \mathbb{N}} \{x \in \mathbb{R}^k : T_{j,\gamma}(x) > 0\}$ has measure zero. \square

The following concept is well known (see, e.g., [60, 7.9]:

Definition A.9 Let $x \in \mathbb{R}^k$. A sequence $(A_n)_{n \in \mathbb{N}}$ of Borel sets in \mathbb{R}^k is said to shrink to x nicely if there exist $\alpha > 0$ and a sequence of balls $B_{r_n}(x)$ with $r_n \rightarrow 0$ such that $A_n \subseteq B_{r_n}(x)$ for all $n \in \mathbb{N}$ and

$$\lambda_k(E_n) \geq \alpha \lambda_k(B_{r_n}(x)).$$

Analogous to [60, 7.10], also in the vector-valued case we have:

Lemma A.10 Let E be a Fréchet space and $\gamma \in \mathcal{L}^1(\mathbb{R}^k, E)$. For each $x \in \mathbb{R}^k$, let $(A_n(x))_{n \in \mathbb{N}}$ be a sequence of Borel sets which shrinks to x nicely. Then

$$\gamma(x) = \lim_{n \rightarrow \infty} \frac{1}{\lambda_k(A_n(x))} \int_{A_n(x)} \gamma d\lambda_k$$

at every Lebesgue point x of γ , and thus λ_k -almost everywhere.

Proof. Let x be a Lebesgue point of γ and $\alpha > 0$ and $(B_{r_n}(x))_{n \in \mathbb{N}}$ be the positive number and the balls associated to the sequence $(E_n(x))_{n \in \mathbb{N}}$. Let q be a continuous seminorm on E . Then

$$\begin{aligned} & \frac{1}{\lambda_k(A_n(x))} \int_{A_n(x)} q(\gamma(y) - \gamma(x)) d\lambda_k(y) \\ & \leq \frac{1}{\alpha \lambda_k(B_{r_n}(x))} \int_{B_{r_n}(x)} q(\gamma(y) - \gamma(x)) d\lambda_k(y). \end{aligned} \quad (110)$$

Since x is a Lebesgue point of γ , the right hand side of (110) tends to 0 as $n \rightarrow \infty$, entailing that also the left hand side tends to 0. \square

Proof of Lemma 1.28 (compare, e.g., [60, Theorem 7.11] for the well-known scalar-valued case). We may assume that $J = \mathbb{R}$ (as we can extend γ by 0 outside J). The weak integrals needed to define η exist by (A.4). To complete the proof, we need only show that η is differentiable with derivative $\gamma(x)$ at each Lebesgue point x of γ (recalling Lemma A.8). Since

$$p(\eta(y) - \eta(x)) \leq \left| \int_x^y p(\gamma(t)) d\lambda_1(t) \right| \rightarrow 0$$

as $y \rightarrow x$ for each continuous seminorm p on E , the map η is continuous. If $x \in \mathbb{R}$ and $(t_{x,n})_{n \in \mathbb{N}}$ is a sequence of real number $t_{x,n} > x$ converging to x , then the sets $A_n(x) := [x, t_{x,n}]$ shrink nicely to x , entailing that

$$\frac{\eta(t_{x,n}) - \eta(x)}{t_{x,n} - x} = \frac{1}{\lambda_1([x, t_{x,n}])} \int_{[x, t_{x,n}]} \gamma d\lambda_1 \rightarrow \gamma(x)$$

at each Lebesgue point x of γ . We have shown that right-sided derivative of η exists at x and equals $\gamma(x)$. Likewise, if $(s_{x,n})_{n \in \mathbb{N}}$ is a sequence of real number $s_n < x$ converging to x , then the sets $[s_{x,n}, x]$ shrink to x nicely and thus

$$\frac{\eta(x) - \eta(s_{x,n})}{x - s_{x,n}} = \frac{1}{\lambda_1([s_{x,n}, x])} \int_{[s_{x,n}, x]} \gamma d\lambda_1 \rightarrow \gamma(x)$$

at each Lebesgue point x . Hence $\gamma(x)$ is also the left-sided derivative of η at x . As a consequence, η is differentiable at x and $\eta'(x) = \gamma(x)$. \square

Proof of Lemma 1.29. Let $\gamma_j \in \mathcal{L}_{rc}^\infty(J, E)$ for $j \in \{1, 2\}$ such that $\eta(t) := \int_{t_0}^t \gamma_1(s) ds = \int_{t_0}^t \gamma_2(s) ds$ for all $t \in J$. Since E is integral complete and hence has the metric CCP, the set $K := \overline{\text{conv}(\text{im}(\gamma_1) \cup \text{im}(\gamma_2))} \subseteq E$ is metrizable and compact (see Lemma 1.21). By Lemma 1.11, there is a metrizable vector topology \mathcal{O}' on $F := \text{span}(K)$ which is coarser than the topology induced by E . As (F, \mathcal{O}') and E induce the same topology on K , we have $\gamma_j \in \mathcal{L}_{rc}^\infty(J, F)$ for $j \in \{1, 2\}$. By the proof of Lemma 1.23, the weak integrals $\int_{t_0}^t \gamma_1(s) ds$ and $\int_{t_0}^t \gamma_2(s) ds$ also exist in F , for all $t \in J$, and coincide with $\eta(t)$. Let \tilde{F} be a completion of F such that $F \subseteq \tilde{F}$ and consider γ_j as an element of $\mathcal{L}_{rc}^\infty(J, \tilde{F})$ for $j \in \{1, 2\}$. Since \tilde{F} is a Fréchet space, we deduce with Lemma 1.28 that $[\gamma_1] = [\gamma_2]$ in $L_{rc}^\infty(J, \tilde{F})$. As a consequence, $[\gamma_1] = [\gamma_2]$ also in $L_{rc}^\infty(J, F)$ and hence also in $L_{rc}^\infty(J, E)$. \square

Lemma A.11 *Let E be a Fréchet space, $J \subseteq \mathbb{R}$ a non-degenerate interval, $\eta: J \rightarrow E$ be a continuous map and A the set of all $t \in J$ such that γ is differentiable at t . Then A is a Borel set in J .*

Proof. Let $p_1 \leq p_2 \leq \dots$ be a sequence of continuous seminorms on E defining its locally convex vector topology. For $j \in \mathbb{N}$ and $\varepsilon > 0$, let $h_{j,\varepsilon}(t)$ be the supremum of the real numbers

$$p_j \left(\frac{\eta(s_1) - \eta(t)}{s_1 - t} - \frac{\eta(s_2) - \eta(t)}{s_2 - t} \right) =: g_{j,s_1,s_2}(t), \quad (111)$$

for $s_1, s_2 \in (J \cap]x - \varepsilon, x + \varepsilon]) \setminus \{t\}$. Note that (111) is a continuous function of s_1 and s_2 . Therefore the same supremum is obtained if we take $s_1, s_2 \in \mathbb{Q} \cap (J \cap]x - \varepsilon, x + \varepsilon]) \setminus \{t\}$ instead. If we set $g_{j,s_1,s_2}(t) := 0$ if $t \in \{s_1, s_2\}$, then

$$g_{j,s_1,s_2}: J \rightarrow [0, \infty[$$

is a measurable function. Since $h_{j,\varepsilon}$ is the supremum of these for countably many (s_1, s_2) as just described, also $h_{j,\varepsilon}$ is measurable. Now $x \in A$ if and only if $(\frac{\gamma(s)-\gamma(t)}{s-t})_{s \neq t}$ is a Cauchy net (using the preorder given by $s_1 \preceq s_2$ if and only if $|s_2 - x| < |s_1 - x|$ to make $J \setminus \{t\}$ a directed set). The latter holds if and only if $h_{j,\varepsilon}(t) \rightarrow 0$ as $\varepsilon \rightarrow 0$, for all $j \in \mathbb{N}$. Since $h_{j,\varepsilon}(t)$ is a decreasing functions of ε , equivalently $h_{j,1/n}(t) \rightarrow 0$ as $n \rightarrow \infty$, for all $j \in \mathbb{N}$. Thus

$$A = \bigcap_{j \in \mathbb{N}} \left\{ t \in J : \lim_{n \rightarrow \infty} h_{j,1/n}(t) = 0 \right\},$$

entailing that A is a Borel set. \square

Proof of Lemma 1.32. Let $\gamma \in \mathcal{R}([a, b], E)$ and $(\eta_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{T}([a, b], E)$ such that $\eta_n \rightarrow \gamma$ uniformly. By Lemma 1.11, there is a sequence $q_1 \leq q_2 \leq \dots$ of continuous seminorms on E such that the vector topology \mathcal{O}' on $F := \overline{\text{span}(\gamma([a, b]))}$ defined by $(q_n|_F)_{n \in \mathbb{N}}$ is Hausdorff. Then E and (F, \mathcal{O}') induce the same topology on $K := \overline{\gamma([a, b])}$. After increasing the seminorms if necessary, we may assume that $2q_n \leq q_{n+1}$ for each $n \in \mathbb{N}$. Hence, for each $x \in K$

$$K \cap B_2^{q_n}(x) \quad \text{with } n \in \mathbb{N} \quad \text{is a basis of neighbourhoods of } x \text{ in } K. \quad (112)$$

For each $n \in \mathbb{N}$, there is $m_n \in \mathbb{N}$ such that

$$\sup_{t \in [a, b]} q_n(\gamma(t) - \eta_k(t)) \leq 1 \quad \text{for all } k \geq m_n. \quad (113)$$

We have $\eta_{m_n}([a, b]) = \{y_{n,1}, \dots, y_{n,\ell_n}\}$ for some $\ell_n \in \mathbb{N}$ and pairwise distinct elements $y_{n,1}, \dots, y_{n,\ell_n} \in E$. By (113), we find $z_{n,j} \in \gamma([a, b])$ for $j \in \{1, \dots, \ell_n\}$ such that $q_n(y_{n,j} - z_{n,j}) \leq 1$. Define $\gamma_n: [a, b] \rightarrow E$ via

$$\gamma_n(t) := z_{n,j} \quad \text{if } \eta_{m_n}(t) = y_{n,j}.$$

Then $\gamma_n \in \mathcal{T}([a, b], E)$, $\gamma_n([a, b]) \subseteq \gamma([a, b])$ and $\sup_{t \in [a, b]} q_n(\gamma(t) - \gamma_n(t)) \leq 2$. By (112), this implies that $\gamma_n \rightarrow \gamma$ uniformly. \square

Proof of Lemma 1.36. Write Q for the map described in (7). Since

$$Q|_{E_1 \times \dots \times E_N} : (x_1, \dots, x_N) \mapsto \left(\sum_{n=1}^N q_n(x_n)^p \right)^{1/p}$$

(resp. $(x_1, \dots, x_N) \mapsto \max\{q_1(x_1), \dots, q_N(x_N)\}$) is a continuous seminorm on $E_1 \times \dots \times E_N$ for each $N \in \mathbb{N}$, and $\bigoplus_{n \in \mathbb{N}} E_n = \varinjlim (E_1 \times \dots \times E_N)$, we deduce that Q is a continuous seminorm on the direct sum. Therefore, the topology defined by the seminorms Q is coarser than the locally convex direct sum topology. That the topologies coincide if $p = \infty$ is well-known and corresponds to the fact that the topology on a countable direct sum is the box topology. If $p < \infty$, then $Q_\infty \leq Q_p$ holds pointwise for the seminorms defined by (7), applied with ∞ (in place of p) and p , respectively. The topology defined by the Q_p is therefore finer than the topology defined by the Q_∞ , which is the locally convex direct sum topology. As it is also coarser, the topologies coincide. \square

Proof of Lemma 1.37. It is well-known that the summation map $\bigoplus_{n \in \mathbb{N}} E_n \rightarrow E$ is a topological quotient map, if we use the locally convex direct sum topology on the left. Since linear quotient maps between locally convex spaces are open maps, the assertion follows from Lemma 1.36. \square

Proof of Lemma 1.39. Let D be a countable dense subset of E and $\{U_n : n \in \mathbb{N}\}$ be a basis of 0-neighbourhoods for F . For each $n \in \mathbb{N}$, there is a continuous seminorm q_n on E such that $F \cap B_1^{q_n}(0) \subseteq U_n$. After replacing q_n with the pointwise maximum $\max\{q_1, \dots, q_n\}$, we may assume that $q_1 \leq q_2 \leq \dots$. Now $\{q_n|_F : n \in \mathbb{N}\}$ is a directed set of seminorms on F defining its topology. For $n, m \in \mathbb{N}$, let

$$D_{n,m} := \{y \in D : B_{1/m}^{q_n}(y) \cap F \neq \emptyset\}.$$

For $y \in D_{n,m}$, pick $z_{n,m,y} \in B_{1/m}^{q_n}(y) \cap F$. Then

$$D_F := \{z_{n,m,y} : n, m \in \mathbb{N}, y \in D_{n,m}\}$$

is a countable subset of F . To see that D_F is dense in F , it suffices to show that

$$B_{2/m}^{q_n}(x) \cap D_F$$

is non-empty for all $x \in F$ and $n, m \in \mathbb{N}$. By density of D in E , we find $y \in B_{1/m}^{q_n}(x) \cap D$. Then $x \in B_{1/m}^{q_n}(y) \cap F$, entailing that $y \in D_{n,m}$. Now $z_{n,m,y} \in D_F$ is an element such that $z_{n,m,y} \in B_{1/m}^{q_n}(y)$, whence

$$q_n(x - z_{n,m,y}) \leq q_n(x - y) + q_n(y - z_{n,m,y}) < \frac{1}{m} + \frac{1}{m} = \frac{2}{m}$$

and thus $z_{n,m,y} \in B_{2/m}^{q_n}(x) \cap D_F$. \square

Proof of 1.40. Let $\gamma_1, \gamma_2 \in \mathcal{L}^p(X, \mu, E)$. Then $\text{span}(\gamma_1(X) \cup \gamma_2(X))$ is a separable vector subspace of E , entailing that

$$\overline{\text{span}(\gamma_1(X) \cup \gamma_2(X))} = \bigcup_{n \in \mathbb{N}} F_n$$

with topological vector subspaces $F_1 \subseteq F_2 \subseteq \dots$ of E which are separable Fréchet spaces. Since F_n is closed in E and hence a Borel set, we deduce that

$$A_n := (\gamma_1)^{-1}(F_n) \cap (\gamma_2)^{-1}(F_n) \in \Sigma$$

for all $n \in \mathbb{N}$. Moreover, $A_1 \subseteq A_2 \subseteq \dots$ with $\bigcup_{n \in \mathbb{N}} A_n = X$. Since F_n is a separable metric space and the addition map $\alpha_n: F_n \times F_n \rightarrow F_n$ is continuous, we see with Lemma A.1 (b) that

$$\gamma_1|_{A_n} + \gamma_2|_{A_n} = \alpha_n \circ (\gamma_1|_{A_n}^{F_n}, \gamma_2|_{A_n}^{F_n})$$

is a measurable map to F_n and hence to E , for each n . Since $(\gamma_1 + \gamma_2)|_{A_n}$ is measurable and $(A_n)_{n \in \mathbb{N}}$ is a countable cover of X by measurable sets, we deduce that $\gamma_1 + \gamma_2$ is measurable. Now $\text{im}(\gamma_1 + \gamma_2) = \bigcup_{n \in \mathbb{N}} (F_n \cap \text{im}(\gamma_1 + \gamma_2))$ is a countable union of separable sets and hence separable. Moreover, $\|\gamma_1 + \gamma_2\|_{\mathcal{L}^p, q} \leq \|q \circ \gamma_1 + q \circ \gamma_2\|_{\mathcal{L}^p} \leq \|q \circ \gamma_1\|_{\mathcal{L}^p} \|q \circ \gamma_2\|_{\mathcal{L}^p} = \|\gamma_1\|_{\mathcal{L}^p, q} + \|\gamma_2\|_{\mathcal{L}^p, q} < \infty$ for each continuous seminorm $q \in P(E)$. Hence $\gamma_1 + \gamma_2 \in \mathcal{L}^p(X, \mu, E)$. If $\gamma \in \mathcal{L}^p(X, \Sigma, \mu)$ and $\|\gamma\|_{\mathcal{L}^p, q} = 0$ for each $q \in P(E)$, pick separable Fréchet spaces $F_1 \subseteq F_2 \subseteq \dots$ in E with

$$\overline{\text{span}(\gamma(X))} = \bigcup_{n \in \mathbb{N}} F_n.$$

Then $\|\gamma|_{\gamma^{-1}(F_n)}\|_{\mathcal{L}^p, q} = 0$ for all $q \in P(F_n)$, entailing that $\gamma|_{\gamma^{-1}(F_n)}: \gamma^{-1}(F_n) \rightarrow F_n$ is 0 outside a set $N_n \subseteq X$ of measure $\mu(N_n) = 0$. Then $\mu(\bigcup_{n \in \mathbb{N}} N_n) = 0$ and $\gamma(x) = 0$ for all $x \in X \setminus \bigcup_{n \in \mathbb{N}} N_n$. Hence $[\gamma] = 0$ and thus $L^p(X, \mu, E)$ is Hausdorff. \square

A.12 Let M be a smooth manifold modelled on a locally convex space X and F be a locally convex space. Recall that a smooth vector bundle over M , with

typical fibre F is a smooth manifold E , together with a surjective smooth map $\pi: E \rightarrow M$ and a vector space structure on

$$E_x := \pi^{-1}(\{x\})$$

for each $x \in M$, such that for each $x \in M$ there exists an open neighbourhood U of x in M and a so-called local trivialization $\theta: E_U \rightarrow U \times F$, with $E_U := \pi^{-1}(U)$. Thus θ is a C^∞ -diffeomorphism such that

$$\text{pr}_1 \circ \theta = \pi$$

(where $\text{pr}_1: U \times F \rightarrow U$, $(x, y) \mapsto x$) and $\text{pr}_2 \circ \theta|_{E_x}: E_x \rightarrow F$ is linear (and hence an isomorphism of topological vector spaces) for each $x \in U$, where $\text{pr}_2: U \times F \rightarrow F$, $(x, y) \mapsto y$.

A.13 If $\pi: E \rightarrow M$ is a smooth vector bundle with typical fibre F A smooth section of E is a smooth map $\sigma: M \rightarrow E$ such that $\pi \circ \sigma = \text{id}_M$. We write $\Gamma(E)$ for the vector space of all smooth sections of E (with pointwise addition and multiplication with scalars). We endow $\Gamma(E)$ with the vector topology making the map

$$\Gamma(E) \rightarrow \prod_{\theta} C^\infty(U_\theta, F), \quad \sigma \mapsto (\text{pr}_2 \circ \theta \circ \sigma|_{U_\theta})$$

a topological embedding onto a closed vector subspace (for $\theta: E_{U_\theta} \rightarrow U_\theta \times F$ ranging through the set of all local trivializations of E).

Now assume that M is paracompact and finite-dimensional. If $K \subseteq M$ is a compact set, we write $\Gamma_K(E)$ for the closed vector subspace of all $\sigma \in \Gamma(E)$ such that $\sigma(x) = 0 \in E_x$ for all $x \in M \setminus K$. We endow

$$\Gamma_c(E) = \bigcup_K \Gamma_K(E)$$

with the locally convex direct limit topology (for K ranging through the set of all compact subsets of M). As the inclusion map $\Gamma_K(E) \rightarrow \Gamma(E)$ for each compact set $K \subseteq M$ is continuous linear, also the inclusion map $\Gamma_c(E) \rightarrow \Gamma(E)$ is continuous. Since $\Gamma(E)$ is Hausdorff, we deduce that $\Gamma_c(E)$ is Hausdorff.

A.14 If $\pi: E \rightarrow M$ is a smooth vector bundle and $U \subseteq M$ is an open subset, then $E|_U := \pi^{-1}(U) \subseteq E$ with the given vector space structure on each fibre and the restriction $\pi|_{E|_U}: E|_U \rightarrow U$. If $K \subseteq M$ is compact and $K \subseteq U$, then the linear map

$$\Gamma_K(E) \rightarrow \Gamma_K(E|_U), \quad \sigma \mapsto \sigma|_U$$

is a bijection and in fact an isomorphism of topological vector spaces (as is clear from the definition of the topologies). Hence also the inverse map

$$\Gamma_K(E|_U) \rightarrow \Gamma_K(E), \quad \sigma \mapsto \tilde{\sigma} \tag{114}$$

(with $\tilde{\sigma}(x) = \sigma(x)$ if $x \in U$, $\tilde{\sigma}(x) = 0$ if $x \in M \setminus K$) is an isomorphism of topological vector spaces.

Lemma A.15 *Let F be a locally convex space and $\pi: E \rightarrow M$ be a smooth vector bundle over a finite-dimensional paracompact smooth manifold M , with typical fibre F . Let $(U_j)_{j \in J}$ be a locally finite cover of M by open, relatively compact subsets $U_j \subseteq M$. Then the map*

$$\Phi: \Gamma_c(E) \rightarrow \bigoplus_{j \in J} \Gamma(E|_{U_j}), \quad \sigma \mapsto (\sigma|_{U_j})_{j \in J}$$

is a topological embedding onto a closed vector subspace which is complemented in the direct sum as a topological vector space.

Proof. It is clear from the definition of the topology that the linear map

$$\Gamma(E) \rightarrow \Gamma(E|_{U_j}), \quad \sigma \mapsto \sigma|_{U_j}$$

is continuous for each $j \in J$. Hence also the restriction to $\Gamma_K(E)$ is continuous linear for each compact set $K \subseteq M$, entailing that the linear map Φ is continuous. Pick a partition of unity $(h_j)_{j \in J}$ subordinate to $(U_j)_{j \in J}$, with compact supports $K_j := \overline{\{x \in M: h_j(x) \neq 0\}}$. Then the multiplication operator

$$\Gamma(E|_{U_j}) \rightarrow \Gamma_{K_j}(E|_{U_j}), \quad \sigma \mapsto h_j \sigma$$

is linear and continuous (see [25] or [35]) and hence also the map

$$\alpha_j: \Gamma(E|_{U_j}) \rightarrow \Gamma(E), \quad \sigma \mapsto (h_j \sigma)^\sim$$

is continuous linear (with notation as in (114)). This map takes its values in $\Gamma_K(E)$, which injects continuously in $\Gamma_c(E)$. We can therefore consider α_j as

a continuous linear map to $\Gamma_c(E)$. By the universal property of the locally convex direct sum, also the map

$$\alpha: \bigoplus_{j \in J} \Gamma(E|_{U_j}) \rightarrow \Gamma_c(E), \quad (\sigma_j)_{j \in J} \mapsto \sum_{j \in J} \alpha_j(\sigma_j)$$

is continuous linear. Now $\alpha \circ \Phi = \text{id}$, entailing that Φ is a topological embedding and

$$\bigoplus_{j \in J} \Gamma(E|_{U_j}) = \text{im}(\Phi) \oplus \ker(\alpha)$$

as a topological vector space. Notably, $\text{im}(\Phi)$ is closed in $\bigoplus_{j \in J} \Gamma(E|_{U_j})$. \square

Remark A.16 The same argument applies if smooth sections are replaced with C^k -sections with $k \in \mathbb{N}_0$. In particular, the map

$$C_c^k(M, E) \rightarrow \bigoplus_{j \in J} C^k(U_j, E), \quad \gamma \mapsto (\gamma|_{U_j})_{j \in J}$$

is a linear topological embedding onto a closed and complemented topological vector subspace for each $k \in \mathbb{N}_0 \cup \{\infty\}$, locally convex space E , paracompact finite-dimensional smooth manifold M and locally finite cover $(U_j)_{j \in J}$ of M by relatively compact, open subsets $U_j \subseteq M$.

Proof of Lemma 1.41. (a) If $S \subseteq \bigoplus_{j \in J} E_j =: E$ is a separable closed vector subspace, let $D \subseteq S$ be a dense countable subset. Since D is countable, there exists a countable subset $J_0 \subseteq J$ such that $D \subseteq \bigoplus_{j \in J_0} E_j$. As the projections $\text{pr}_i: E \rightarrow E_i$, $(x_j)_{j \in J} \mapsto x_i$ are continuous linear, $F := \bigoplus_{j \in J_0} E_j = \bigcap_{i \in J \setminus J_0} \ker \text{pr}_i$ is a closed vector subspace of E with $D \subseteq F$ and hence $S \subseteq F$. Now $S_i := \overline{\text{pr}_i(S)} \subseteq E_i$ is a separable closed vector subspace for each $i \in J_0$ and hence $S_i = \bigcup_{k \in \mathbb{N}} F_{i,k}$ with separable Fréchet subspaces $F_{i,1} \subseteq F_{i,2} \subseteq \dots$ of E_i , as E_i has the (FEP). If J_0 is infinite, enumerate $J_0 = \{j_1, j_2, \dots\}$; if J_0 has a finite number, m , of elements, write $J_0 = \{j_1, \dots, j_m\}$. Then

$$Y := \bigoplus_{j \in J_0} S_j = \bigcap_{i \in J \setminus J_0} \ker(\text{pr}_i) \cap \bigcap_{j \in J_0} (\text{pr}_j)^{-1}(S_j)$$

is a closed vector subspace which contains D and thus $S \subseteq Y = \bigcup_{n \in \mathbb{N}} Y_n$ with the separable Fréchet spaces $Y_n := F_{j_1,n} \times \dots \times F_{j_m,n} \subseteq \bigoplus_{j \in J} E_j$ (if J_0 is infinite), resp.,

$$Y_n := F_{j_1,n} \times \dots \times F_{j_m,n}$$

(if J_0 is finite). Then $(Y_n \cap S)_{n \in \mathbb{N}}$ is an ascending sequence of separable Fréchet subspaces of $\bigoplus_{j \in J} E_j$ with $S = \bigcup_{n \in \mathbb{N}} (Y_n \cap S)$.

If $\gamma = (\gamma_j)_{j \in J} \in \mathcal{L}^p(X, \mu, E)$, then $S := \overline{\text{span}(\gamma(X))}$ is a closed vector subspace of E and $S \subseteq \bigoplus_{j \in J_0} E_j$ with some countable subset $J_0 \subseteq J$. Then

$$J_1 := \{j \in J_0 : \mu((\gamma_j)^{-1}(E_j \setminus \{0\})) > 0\}$$

is a finite set, from which $L^p(X, \mu, E) = \bigoplus_{j \in J} L^p(X, \mu, E_j)$ follows. In fact, if J_1 was infinite, we could choose a bijection $\mathbb{N} \rightarrow J$, $n \rightarrow j_n$. Since $[\gamma_{j_n}] \neq 0$, we would find a continuous seminorm q_{j_n} on E_{j_n} such that $\|\gamma_{j_n}\|_{\mathcal{L}^p, q_{j_n}} > 0$; after replacing q_{j_n} with a multiple, we may assume that

$$\|\gamma_{j_n}\|_{\mathcal{L}^p, q_{j_n}} \geq n$$

for all $n \in \mathbb{N}$. For $j \in J \setminus J_1$, choose any $q_j \in P(E_j)$. Then

$$q: E \rightarrow [0, \infty[, \quad q((y_j)_{j \in J}) := \sum_{j \in J} q_j(y_j)$$

is a continuous seminorm on E . For each $n \in \mathbb{N}$, we have

$$\|\gamma\|_{\mathcal{L}^p, q} = \sqrt[p]{\int_X q(\gamma(x))^p d\mu(x)} \geq \sqrt[p]{\int_X q_{j_n}(\gamma(x_{j_n}))^p d\mu(x)} = \|\gamma_{j_n}\|_{\mathcal{L}^p, q_{j_n}} \geq n.$$

Hence $\|\gamma\|_{\mathcal{L}^p, q} = \infty$, contradicting $\gamma \in \mathcal{L}^p(X, \mu, E)$.

As each inclusion map

$$L^p(X, \mu, E_j) \rightarrow L^p(X, \mu, E)$$

is continuous and linear, the universal property of the locally convex direct sums provides the continuity of the linear summation map

$$\Sigma: \bigoplus_{j \in J} L^p(X, \mu, E_j) \rightarrow L^p(X, \mu, E), \quad (\gamma_j)_{j \in J} \mapsto \sum_{j \in J} \gamma_j.$$

If J is countable or $p = 1$, let us show that also Σ^{-1} is continuous. To this end, let Q be a continuous seminorm on $\bigoplus_{j \in J} L^p(X, \mu, E_j)$. After increasing Q , we may assume that

$$Q((\gamma_j)_{j \in J}) = \sum_{j \in J} Q_j(\gamma_j)$$

with continuous seminorm Q_j on $L^1(X, \mu, E_j)$. After increasing each Q_j if necessary, we may assume that

$$Q_j = \|\cdot\|_{L^1, q_j}$$

for a continuous seminorm q_j on E_j , for each $j \in J$. Then $q: E \rightarrow [0, \infty[$,

$$q((v_j)_{j \in J}) := \sum_{j \in J} q_j(v_j)$$

is a continuous seminorm on E . If $[\gamma] \in L^1(X, \mu, E)$, we may assume that the representative γ has been chosen in $\bigoplus_{j \in J} \mathcal{L}^1(X, \mu, E_j)$ (as just shown). Thus $\gamma = \sum_{j \in F} \gamma_j$ for some finite subset $F \subseteq J$ and suitable $\gamma_j \in \mathcal{L}^1(X, \mu, E_j)$. Thus

$$\begin{aligned} Q(\Sigma^{-1}([\gamma])) &= \sum_{j \in F} \|\gamma_j\|_{\mathcal{L}^1, q_j} = \sum_{j \in F} \int_X q_j(\gamma_j(x)) d\mu(x) \\ &= \int_X \sum_{j \in F} q_j(\gamma_j(x)) d\mu(x) = \int_X q((\gamma_j(x))_{j \in J}) d\mu(x) \\ &= \int_X q(\gamma(x)) d\mu(x) = \|[\gamma]\|_{L^1, q} \leq \|[\gamma]\|_{L^1, q} \end{aligned}$$

and thus Σ^{-1} is continuous.

(b) If S is a closed vector subspace of F , then S is also closed in E and thus $S = \bigcup_{n \in \mathbb{N}} F_n$ with an ascending sequence $(F_n)_{n \in \mathbb{N}}$ of separable Fréchet spaces. Hence F has the (FEP).

(c) By Proposition 9 in [9, Chapter II, §4, no.6], E induces the given topology on each F_n . If $S \subseteq E$ is a closed vector subspace, then $S = \bigcup_{n \in \mathbb{N}} (S \cap F_n)$ where $S \cap F_n$ is a Fréchet space for each $n \in \mathbb{N}$.

(d) Let $(U_j)_{j \in J}$ be a locally finite cover of M by open, relatively compact subsets $U_j \subseteq M$ such that there exists a trivialization $\theta_j: E|_{U_j} \rightarrow U_j \times F$. Then $\Gamma(E|_{U_j}) \cong C^\infty(U_j, F)$ is a Fréchet space for each $j \in J$ (cf. [25] or [35]). Since, by Lemma A.15, there exists a linear topological embedding

$$\Phi: \Gamma_c(E) \rightarrow \bigoplus_{j \in J} \Gamma(E|_{U_j})$$

with closed image, we deduce with (a) and (b) that $\Gamma_c(E)$ has the (FEP) and with (a) that, for each $\gamma \in \mathcal{L}^p(X, \mu, E)$, there exists a finite subset $J_0 \subseteq J$

and $A \in \Sigma$ with $\mu(X \setminus A) = 0$ and

$$(\Phi \circ \gamma)(A) \subseteq \bigoplus_{j \in J_0} \Gamma(E|_{U_j}).$$

Let $\mathbf{1}_A: X \rightarrow \{0, 1\}$ be the characteristic function (indicator function) of A . Then $K := \overline{\bigcup_{j \in J_0} U_j}$ is a compact subset of M and

$$[\gamma] = [\mathbf{1}_A \gamma] \in L^p(X, \mu, \Gamma_K(E)).$$

(e) This is a special case of (d). \square

Proof of Lemma 1.43. If $\gamma \in \mathcal{L}^1(X, \mu, E)$, then $S := \overline{\text{span}(\gamma(X))}$ is a separable closed vector subspaces of E and thus $S = \bigcup_{n \in \mathbb{N}} F_n$ with separable Fréchet spaces $F_1 \subseteq F_2 \subseteq \cdots$ in E (as E has the (FEP)). Since F_n is complete, hence closed in E and thus Borel, we have $A_n := \gamma^{-1}(F_n) \in \Sigma$ for each $n \in \mathbb{N}$, $A_1 \subseteq A_2 \subseteq \cdots$ and $X = \bigcup_{n \in \mathbb{N}} A_n$. Since $\gamma|_{A_n} \in \mathcal{L}^1(A_n, \mu|_{A_n}, F_n)$, the weak integral

$$z_n := \int_{A_n} \gamma|_{A_n} d(\mu|_{A_n}) = \int_X \mathbf{1}_{A_n} \gamma d\mu$$

exists in F_n (and hence in E), for each $n \in \mathbb{N}$. We claim that $(z_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in E . If this is true, then

$$z := \lim_{n \rightarrow \infty} z_n$$

exists in E since E is assumed sequentially complete. For each $\lambda \in E'$, we have

$$\lambda(z) = \lim_{n \rightarrow \infty} \lambda(z_n) = \lim_{n \rightarrow \infty} \int_X \mathbf{1}_{A_n} (\lambda \circ \gamma) d\mu = \int_X (\lambda \circ \gamma) d\mu$$

by dominated convergence (with $|\lambda \circ \gamma|$ as a majorant) and thus $z = \int_X \gamma d\mu$.

To prove the claim, let $q \in P(E)$. Since $q \circ \gamma \in \mathcal{L}^1(X, \mu, \mathbb{R})$, we have

$$\int_X \mathbf{1}_{A_n} (q \circ \gamma) d\mu \rightarrow \int_X (q \circ \gamma) d\mu$$

as $n \rightarrow \infty$ by dominated convergence, entailing that $(\int_X \mathbf{1}_{A_n} (q \circ \gamma) d\mu)_{n \in \mathbb{N}}$ is a Cauchy sequence. Hence, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all

$m \geq n \geq N$:

$$\begin{aligned}
q(z_m - z_n) &= q \left(\int_X (\mathbf{1}_{A_m} - \mathbf{1}_{A_n}) \gamma \, d\mu \right) \\
&\leq \int_X (\mathbf{1}_{A_m} - \mathbf{1}_{A_n}) (q \circ \gamma) \, d\mu \\
&= \int_X \mathbf{1}_{A_m} (q \circ \gamma) \, d\mu - \int_X \mathbf{1}_{A_n} (q \circ \gamma) \, d\mu < \varepsilon
\end{aligned}$$

Thus $(z_n)_{n \in \mathbb{N}}$ indeed is a Cauchy sequence. \square

Proof of Lemma 1.44. Let $\gamma \in \mathcal{L}^p(X, \mu, E)$. We have $\overline{\text{span } \gamma(X)} = \bigcup_{n \in \mathbb{N}} F_n$ for suitable vector subspaces $F_1 \subseteq F_2 \subseteq \dots$ of E which are separable Fréchet spaces and hence closed. Then $A_n := \gamma^{-1}(F_n)$ are Borel sets with $A_1 \subseteq A_2 \subseteq \dots$ and $X = \bigcup_{n \in \mathbb{N}} A_n$. For each continuous seminorm q on E , we have

$$\|\gamma - \gamma \cdot \mathbf{1}_{A_n}\|_{\mathcal{L}^p, q} = \left(\int_X (q(\gamma(x)))^p (1 - \mathbf{1}_{A_n}(x)) \, d\mu(x) \right)^{1/p} \rightarrow 0$$

by dominated convergence. Hence $\gamma \cdot \mathbf{1}_{A_n} \rightarrow \gamma$. Hence, if each $\gamma \mathbf{1}_{A_n}$ is in the closure of $\mathcal{F}(X, \mu, E) \cap \mathcal{L}^p(X, \mu, E)$ (or the subset specified in (8)), then also γ is in the closure. After replacing γ with $\gamma \mathbf{1}_{A_n}$ and E with F_n , we may therefore assume that E is a separable Fréchet space. This case was already settled. \square

Proof of Lemma 1.46. By Lemma 1.43, the integrals needed to define η exist in E . To see that η is continuous on the left at each $t \in J$ (a similar argument shows that η is continuous on the right at t). Let $q \in P(E)$. For each $t_1 \in J$ such that $t_1 \leq t$, we have that

$$q(\eta(t) - \eta(t_1)) = q \left(\int_{t_1}^t \gamma(s) \, ds \right) \leq \int_{t_1}^t (q \circ \gamma)(s) \, ds = \int_J \mathbf{1}_{[t_1, t]}(s) (q \circ \gamma)(s) \, ds \rightarrow 0$$

as $t_1 \rightarrow t$, by dominated convergence.

To see that the linear map $L^1(J, E) \rightarrow C(J, E)$, $[\gamma] \mapsto \eta_\gamma$ (with $\eta_\gamma(t) := \int_{t_0}^t \gamma(s) \, d\mu(s)$) is injective, let $\gamma \in \mathcal{L}^1(J, E)$ such that $[\gamma] \neq 0$. Then $\|\gamma\|_{\mathcal{L}^1, q} \neq 0$ for some $q \in P(E)$. Let $\pi_q: E \rightarrow \tilde{E}_q$ and the norm $\|\cdot\|_q$ on \tilde{E}_q be as in 1.45. Then $\|\pi_q \circ \gamma\|_{\mathcal{L}^1, \|\cdot\|_q} = \|\gamma\|_{\mathcal{L}^1, q} \neq 0$, whence $[\pi_q \circ \gamma] \neq 0$ in $L^1(J, \tilde{E}_q)$. Since

$$(\pi_q \circ \eta_\gamma)(t) = \int_{t_0}^t (\pi_q \circ \gamma)(s) \, d\mu(s)$$

for each $t \in J$, Lemma 1.28 shows that $\pi_q \circ \eta_\gamma \neq 0$. Hence $\eta_\gamma \neq 0$. \square

Proof of 1.47. If $[\gamma] = [\gamma_1]$ with γ_1 in $\mathcal{L}^1([a, b], F)$ or $L_{rc}^\infty([a, b], F)$, then γ can be replaced with γ_1 in the definition of η . But then the weak integral defining $\eta(t)$ can be formed in F , and coincide with those in E . Thus $\eta(t) \in F$ for all $t \in [a, b]$. If, conversely,

$$\lambda_1(\{t \in [a, b]: \gamma(t) \in E \setminus F\}) > 0,$$

then

$$[q \circ \gamma] \neq 0$$

in $L^1([a, b], E/F)$ (resp., $L_{rc}^\infty([a, b], E/F)$), where

$$q: E \rightarrow E/F, \quad x \mapsto x + F$$

is the canonical quotient map. Now

$$(q \circ \eta)(t) = \int_a^t q(\gamma(s)) ds$$

for all $t \in [a, b]$. If E is a Fréchet space, then also E/F is a Fréchet space and the uniqueness assertion in Lemma 1.28 shows that $q \circ \eta \neq 0$ and thus $\eta([a, b]) \not\subseteq F$. If $\gamma \in \mathcal{L}_{rc}^\infty([a, b], E)$, then $q \circ \gamma \in \mathcal{L}_{rc}^\infty([a, b], E/F)$. If $q \circ \eta$ would vanish, it would also be a primitive of the constant curve with value 0 and the contradiction $[q \circ \gamma] = [0]$ would follow analogously to the uniqueness part of Lemma 1.29 (whose proof only requires the existence of the weak integrals at hand, not integral completeness). Thus $q \circ \eta \neq 0$ and thus $\eta([a, b]) \not\subseteq F$.

Now assume that E is a strict (LF)-space, say $E = \varinjlim E_n$ as a locally convex space with an ascending sequence $E_1 \subseteq E_2 \subseteq \dots$ of Fréchet spaces such that E_{n+1} induces the given topology on E_n for each n . If a vector subspace $F \subseteq E$ is a Fréchet space in the induced topology, then $F \subseteq E_N$ for some $N \in \mathbb{N}$ as a consequence of the Grothendieck Factorization Theorem [48, 24.33] (or simply using that the locally convex direct limit $E = \bigcup_{n \in \mathbb{N}} E_n$ is regular, whence F cannot contain a zero-sequence leaving each E_n). By Lemma 1.41 (c), we may assume that $\gamma \in \mathcal{L}^1([a, b], E_n)$ for some $n \in \mathbb{N}$; we may assume that $n \geq N$. Then F is a closed vector subspace of the Fréchet space E_n and hence we can replace γ by an element of $\mathcal{L}^1([a, b], F)$ by the special case of the lemma for Fréchet spaces already discussed. \square

Proof of 1.33. To prove the assertion, let $(q_m)_{m \in \mathbb{N}}$ be a sequence in $P(E)$ defining the locally convex vector topology on E . If γ is in the closure,

then we find a sequence $\gamma_n \in \mathcal{T}([a, b], E)$ such that $\|\gamma - \gamma_n\|_{\mathcal{L}^\infty, q_m} \rightarrow 0$ for all $m \in \mathbb{N}$. For each m and n , there is a measurable set $A_{m,n} \subseteq X$ with $\mu(A_{m,n}) = 0$ such that $\|\gamma - \gamma_n\|_{\mathcal{L}^\infty, q_m} = \sup_{x \in X \setminus A_{m,n}} q_m(\gamma(x) - \gamma_n(x))$. Then also $A := \bigcup_{n,m \in \mathbb{N}} A_{m,n}$ is measurable and $\mu(A) = 0$. Define $\eta_n(x) := \gamma_n(x)$ if $x \in X \setminus A$, $\eta_n(x) := \gamma(x)$ if $x \in A$. Then $[\eta_n] = [\gamma_n]$ in $L_{rc}^\infty([a, b], E)$ and $\eta_n \rightarrow \gamma$ uniformly. \square

Proof of 1.35. In fact, the projections $\pi_j: E_1 \times E_2 \rightarrow E_j$ onto the components are continuous linear for $j \in \{1, 2\}$ and also the mappings

$$\lambda_1: E_1 \rightarrow E_1 \times E_2, \quad x \mapsto (x, 0)$$

and $\lambda_2: E_2 \rightarrow E_1 \times E_2, y \mapsto (0, y)$ are continuous linear. Thus

$$\Phi := (L^p(X, \mu, \pi_1), L^p(X, \mu, \pi_2)): L^p(X, \mu, E_1 \times E_2) \rightarrow L^p(X, \mu, E_1) \times L^p(X, \mu, E_2)$$

is continuous linear and also

$$\Psi: L^p(X, \mu, E_1) \times L^p(X, \mu, E_2) \rightarrow L^p(X, \mu, E_1 \times E_2),$$

$\Psi([\gamma_1], [\gamma_2]) := L^p(X, \mu, \lambda_1)([\gamma_1]) + L^p(X, \mu, \lambda_2)([\gamma_2])$ is continuous linear. Since $\Psi \circ \Phi$ is the identity map on $L^p(X, \mu, E_1 \times E_2)$ and $\Phi \circ \Psi$ is the identity map on $L^p(X, \mu, E_1) \times L^p(X, \mu, E_2)$, we see that Φ is an isomorphism of topological vector spaces with inverse Ψ . \square

Proof of Lemma 1.57. The set $U^{[1]} := \{(x, y, t) \in U \times E \times \mathbb{R}: x + ty \in U\}$ is open in $E \times E \times \mathbb{R}$. Since $f: U \rightarrow F$ is C^1 , the map

$$f^{[1]}: U^{[1]} \rightarrow F, \quad (x, y, t) \mapsto \begin{cases} \frac{f(x+ty) - f(x)}{t} & \text{if } t \neq 0; \\ df(x, y) & \text{if } t = 0 \end{cases}$$

is continuous (see [5] or [38]). For $t \in \mathbb{R} \setminus \{0\}$ such that $t_0 + t \in J$, we have

$$\begin{aligned} \frac{f(\gamma(t_0 + t)) - f(\gamma(t_0))}{t} &= \frac{f\left(\gamma(t_0) + t \frac{\gamma(t_0 + t) - \gamma(t_0)}{t}\right) - f(\gamma(t_0))}{t} \\ &= f^{[1]}\left(\gamma(t_0), \frac{\gamma(t_0 + t) - \gamma(t_0)}{t}, t\right), \end{aligned}$$

which converges to $f^{[1]}(\gamma(t_0), \gamma'(t_0), 0) = df(\gamma(t_0), \gamma'(t_0))$ as $t \rightarrow 0$. Thus $(f \circ \gamma)'(t_0) = df(\gamma(t_0), \gamma'(t_0))$. \square

Proof of Lemma 1.59. Since $df: V \times E \rightarrow F$ is continuous and $df(x, 0) = 0 \in B_1^p(0)$, there is a convex open neighbourhood $V_1 \subseteq V$ of x and an open

0-neighbourhood $W \subseteq E$ such that $df(V_1 \times W) \subseteq B_1^p(0)$. We may assume that $W = B_1^q(0)$ for some $q \in P(E)$. Then

$$p(df(y, z)) \leq q(z) \quad \text{for all } y \in V_1 \text{ and } z \in E.$$

For all $y, z \in V_1$, we obtain

$$\begin{aligned} p(f(z) - f(y)) &= q \left(\int_0^1 df(y + t(z - y), z - y) dt \right) \\ &\leq \int_0^1 p(df(y + t(z - y), z - y)) dt \leq q(z - y), \end{aligned}$$

as desired. \square

Proof of Lemma 1.61. The map $d_1f: V \times E_2 \times E_1 \rightarrow F$, $d_1f(x, v, h) := (D_{(h,0)}f)(x, v)$ is continuous and $d_1(K \times \{0\} \times \{0\}) + \{0\} \subseteq B_1^p(0)$. Using the Wallace Lemma (see 1.1), we find an open subset $U \subseteq V$ such that $K \subseteq U$ and continuous seminorms $q_1 \in P(E_1)$ and $q_2 \in P(E_2)$ such that

$$d_1f(u, v, h) \in B_1^p(0) \quad \text{for all } u \in U, h \in B_1^{q_1}(0) \text{ and } v \in B_1^{q_2}(0).$$

As a consequence,

$$p(d_1f(u, v, h)) \leq q_1(h)q_2(v) \quad \text{for all } u \in U, h \in E_1 \text{ and } v \in E_2.$$

Since $f(K \times \{0\}) = \{0\} \subseteq B_1^p(0)$, the Wallace Lemma shows that we may assume that, moreover,

$$f(U \times B_1^{q_2}(0)) \subseteq B_1^p(0)$$

after shrinking U and increasing q_2 . Thus

$$p(f(u, v)) \leq q_2(v) \quad \text{for all } u \in U \text{ and } v \in E_2. \quad (115)$$

After increasing q_1 if necessary, we may assume that $K + B_2^{q_1}(0) \subseteq U$. Let $y, z \in K + B_1^{q_1}(0)$. If $q_1(y - z) < 1$, choose $x \in K$ such that $y \in B_1^{q_1}(x)$. Then $z \in B_2^{q_1}(x)$, by the triangle inequality. For all $v, w \in E_2$, we deduce that

$$\begin{aligned} f(z, v) - f(y, w) &= f(z, v - w) + f(z, w) - f(y, w) \\ &= f(z, v - w) + \int_0^1 d_1f(y + t(z - y), w, z - y) dt. \end{aligned}$$

Hence

$$\begin{aligned}
p(f(z, v) - f(y, w)) &\leq p(f(z, v - w)) \\
&\quad + \sup_{t \in [0, 1]} \underbrace{p(d_1 f(y + t(z - y), w, z - y))}_{\leq q_1(z - y)q_2(w)} \\
&\leq q_2(v - w) + q_1(z - y)q_2(w).
\end{aligned}$$

If $q_1(y - z) \geq 1$, we estimate with (115)

$$\begin{aligned}
p(f(z, v) - f(y, w)) &\leq p(f(z, v - w)) + p(f(z, w)) + p(f(y, w)) \\
&\leq q_2(v - w) + 2q_2(w) \leq q_2(v - w) + 2q_1(y - z)q_2(w).
\end{aligned}$$

We therefore always have (9) if we replace q_1 with $2q_1$. \square

Proof of Lemma 1.62. Using Lemma 1.59, we find a continuous seminorm $q \in P(E)$ such that $B_2^q(x) \subseteq V$ and

$$p(f(z) - f(y)) \leq q(z - y) \quad \text{for all } z, y \in B_2^q(x).$$

Thus $p(f(z) - f(y)) = 0$ for all $z, y \in B_2^q(x)$ such that $q(z - y) = 0$. Equivalently, $\pi_p(f(z)) = \pi_p(f(y))$ for all $z, y \in B_2^q(x)$ such that $\pi_q(z) = \pi_q(y)$. We therefore get a well-defined map

$$\tilde{f}: \pi_q(B_2^q(x)) \rightarrow \tilde{F}_p, \quad \pi_q(y) \mapsto \pi_p(f(y))$$

on the open ball $\pi_q(B_2^q(x)) = \{v \in E_q: \|v - \pi_q(x)\|_q < 2\}$ in the normed space E_q . The map \tilde{f} is Lipschitz continuous with Lipschitz constant 1, as

$$\begin{aligned}
\|\tilde{f}(\pi_q(z)) - \tilde{f}(\pi_q(y))\|_p &= \|\pi_p(f(z) - f(y))\|_p \\
&= p(f(z) - f(y)) \leq q(z - y) = \|\pi_q(z) - \pi_q(y)\|_q
\end{aligned}$$

for all $z, y \in B_2^q(0)$. In particular, \tilde{f} is uniformly continuous and hence extends uniquely to a continuous (and indeed Lipschitz continuous) map

$$g: B_2^{\|\cdot\|_q}(x) \rightarrow \tilde{F}_p$$

on the open ball $B_2^{\|\cdot\|_q}(x) \subseteq \tilde{E}_q$. Since $df: V \times E \rightarrow F$ is C^1 , we can repeat the reasoning. After increasing q if necessary, we may assume that there is a continuous map

$$h: B_2^{\|\cdot\|_q}(x) \times B_2^{\|\cdot\|_q}(0) \rightarrow \tilde{F}_p$$

such that $h(\pi_q(y), \pi_q(z)) = \pi_p(df(y, z))$ for all $(y, z) \in B_2^q(x) \times B_2^q(0)$. Then

$$h(v, rw) = rh(v, w) \quad \text{for all } (v, w) \in B_2^{\|\cdot\|_q}(x) \times B_2^{\|\cdot\|_q}(0) \text{ and } r \in]0, 1]$$

as h is continuous and equality holds for all (v, w) in the dense subset $\pi_q(B_2^q(x)) \times \pi_q(B_2^q(0))$. Therefore

$$H: B_2^{\|\cdot\|_q}(x) \times \tilde{E}_q \rightarrow \tilde{F}_p, \quad H(v, w) := nh\left(v, \frac{1}{n}w\right)$$

for all $n \in \mathbb{N}$, $v \in B_2^{\|\cdot\|_q}(x)$ and $w \in B_{2n}^{\|\cdot\|_q}(0)$ is well-defined. Since H is continuous on the open sets $B_2^{\|\cdot\|_q}(x) \times B_{2n}^{\|\cdot\|_q}(x)$ for $n \in \mathbb{N}$ which cover $B_2^{\|\cdot\|_q}(x) \times \tilde{E}_q$, the map H is continuous. Now, if $y \in B_2^q(x)$ and $y \in B_{2n}^q(0)$, then

$$\begin{aligned} H(\pi_q(y), \pi_q(z)) &= nh\left(\pi_q(y), \frac{1}{n}\pi_q(z)\right) = nh\left(\pi_q(y), \pi_q\left(\frac{1}{n}z\right)\right) \\ &= ndf\left(y, \frac{1}{n}z\right) = df(y, z). \end{aligned}$$

Since $\pi_q(B_2^q(x)) \times E_q$ is dense in $B_2^{\|\cdot\|_q}(x) \times \tilde{E}_q$, we deduce that

$$H(v, \cdot): \tilde{E}_q \rightarrow \tilde{F}_p$$

is linear for each $v \in B_1^{\|\cdot\|_q}(x)$. The parameter-dependent integral

$$g_1: B_1^{\|\cdot\|_q}(x) \times B_1^{\|\cdot\|_q}(0) \times]-1, 1[\rightarrow \tilde{F}_p, \quad a(v, w, t) := \int_0^1 H(v + stw, w) ds$$

is continuous, by 1.18. For all $y \in B_1^q(x)$, $z \in B_1^q(0)$ and $t \in]-1, 1[\setminus \{0\}$, we have

$$\begin{aligned} g_1(\pi_q(y), \pi_q(z)) &= \int_0^1 H(\pi_q(y + stz), \pi_q(z)) ds = \int_0^1 df(y + stz, z) ds \\ &= \frac{1}{t}(f(y + tz) - f(y)) = \frac{1}{t}(g(\pi_q(y) + t\pi_q(z)) - g(\pi_q(y))). \end{aligned}$$

Hence

$$g_1(v, w, t) = \frac{1}{t}(v + tw) - g(v)$$

for all $(x, y, t) \in B_1^{\|\cdot\|_q}(x) \times B_1^{\|\cdot\|_q}(0) \times]-1, 1[\setminus \{0\}$, as both sides of the equation are continuous functions of $(x, y, t) \in B_1^{\|\cdot\|_q}(x) \times B_1^{\|\cdot\|_q}(0) \times]-1, 1[\setminus \{0\}$ which agree on the dense subset $\pi_q(B^q(x)) \times \pi_q(B_1^q(0))$. Abbreviate $U := B_1^{\|\cdot\|_q}(\pi_q(x))$. By the preceding, the map

$$(g|_U)^{[1]}: U^{[1]} = U^{]1[} \cup (U \times B_1^{\|\cdot\|_q}(0) \times]-1, 1]) \rightarrow \widetilde{F}_p,$$

$$(v, w, t) \mapsto \begin{cases} g_1(v, w, t) & \text{if } (v, w, t) \in U \times B_1^{\|\cdot\|_q}(0) \times]-1, 1[; \\ \frac{1}{t}(g(v + tw) - g(v)) & \text{if } (v, w, t) \in U^{]1[} \end{cases}$$

is well-defined, and it is continuous as it is piecewise defined and continuous on the two open pieces. From 1.51, we deduce that $g|_U$ is C^1 , with $dg = g^{[1]}(\cdot, 0) = H|_{U \times \widetilde{E}_q}$. By construction, $\pi_p \circ f|_{B_1^q(0)} = g|_U \circ \pi_q|_{B_1^q(0)}^U$. \square

B Details for Section 7

We study the compatibility of Lebesgue spaces with countable projective limits.

Lemma B.1 *Let $((G_n)_{n \in \mathbb{N}}, (\phi_{n,m})_{n \leq m})$ be a projective system of metrizable, complete topological groups G_n and continuous homomorphisms $\phi_{n,m}: G_m \rightarrow G_n$ such that $\phi_{n,m}(G_m)$ is dense in G_n , for all $n, m \in \mathbb{N}$ with $n \leq m$. Let $G := \varprojlim G_n$ be a projective limit. Then also each limit map $\phi_n: G \rightarrow G_n$ has dense image.*

Proof. It suffices to show that ϕ_1 has dense image. Let $x_1 \in G_1$ and $V \subseteq G_1$ be an open identity neighbourhood. We construct an element $y \in G$ such that $\phi_1(y) \in x_1 \overline{V}$, where \overline{V} is the closure of V . There is a sequence $(U_{1,k})_{k \in \mathbb{N}_0}$ of open identity neighbourhoods $U_{1,k} \subseteq G_1$ such that $U_{1,0} = V$ and $U_{1,k} U_{1,k} \subseteq U_{1,k-1}$ for all $k \in \mathbb{N}$. Thus

$$U_{1,k} U_{1,k+1} \cdots U_{1,\ell-1} U_{1,\ell} \subseteq U_{1,\ell} \subseteq U_{1,k-1} \quad \text{for all } k \in \mathbb{N} \text{ and } \ell > k.$$

Recursively, we choose sequences $(U_{n,k})_{k \in \mathbb{N}_0}$ of open identity neighbourhoods in G_n for $n \in \{2, 3, \dots\}$ such that

$$\phi_{n-1,n}(U_{n,k}) \subseteq U_{n-1,k} \quad \text{for all } k \in \mathbb{N}_0$$

and $U_{n,k} U_{n,k} \subseteq U_{n,k-1}$ for all $k \in \mathbb{N}$. The open set $x_1 U_{1,1}$ contains $\phi_{1,2}(x_2)$ for some $x_2 \in G_2$. Recursively, we find $x_n \in G_n$ for $n \in \{2, 3, \dots\}$ such that

$\phi_{n-1,n}(x_n) \in x_{n-1}U_{n-1,n-1}$. Let $k \in \mathbb{N}$ and $N \in \mathbb{N}$ with $N \geq k$. For all $n, m \in \mathbb{N}$ with $m > n \geq N$, we then have

$$\phi_{n,m}(x_m) \in x_n U_{n,n} U_{n,n+1} \cdots U_{n,m-2} U_{n,m-1} \subseteq x_n U_{n,n-1},$$

entailing that $x_n^{-1} \phi_{n,m}(x_m) \in U_{n,n-1}$ and hence

$$\phi_{k,n}(x_n)^{-1} \phi_{k,m}(x_m) \in U_{k,n-1} \subseteq U_{k,N-1} \quad \text{for all } m > n \geq N. \quad (116)$$

Hence $(\phi_{k,n}(x_n))_{n \geq k}$ is a Cauchy sequence in G_k and thus convergent, to $y_k \in G_k$, say. Letting $m \rightarrow \infty$ in (116), we find that $(\phi_{k,N}(\phi_{N,n}(x_n)))^{-1} y_k = \phi_{k,n}(x_n)^{-1} y_k \in \overline{U_{k,N-1}}$. Letting $n \rightarrow \infty$, we see that $(\phi_{k,N}(y_N))^{-1} y_k \in \overline{U_{k,N-1}}$ and thus

$$\lim_{N \rightarrow \infty} \phi_{k,N}(y_N) = y_k \quad \text{for each } k \in \mathbb{N}. \quad (117)$$

Consider $z_k := (\phi_{1,k}(y_k), \phi_{2,k}(y_k), \dots, \phi_{k-1,k}(y_k), y_k, e, e, \dots) \in \prod_{k \in \mathbb{N}} G_k$. By (117), the sequence $(z_k)_{k \in \mathbb{N}}$ converges to some $z \in \varprojlim G_k = G$. If $\ell \in \mathbb{N}$, taking $k = N = 1$ in (116) and let $m \rightarrow \infty$. using that $\phi_{1,n}(x_m) = \phi_{1,\ell}(\phi_{\ell,m}(x_m))$ if $m \geq \ell$, we see that $(x_1)^{-1} \phi_{1,\ell}(y_\ell) \in \overline{U_{1,0}} \subseteq \overline{V}$. Since $\phi_{1,\ell}(y_\ell)$ converges to $\phi_1(z)$ as $\ell \rightarrow \infty$, we deduce that $\phi_1(z) \in x_1 \overline{V}$. \square

Proof of Lemma 7.1. After passing to an isomorphic locally convex space, we may assume that E is the vector subspace

$$\lim_{\leftarrow} E_n = \{(x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} E_n : (\forall n \leq m) \phi_{n,m}(x_m) = x_n\}$$

of the direct product, with the projection onto the component E_n as the limit map ϕ_n . We realize $\lim_{\leftarrow} L^p(X, \mu, E_n)$ as the vector subspace

$$\{([\gamma_n])_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} L^p(X, \mu, E_n) : (\forall n \leq m) [\gamma_n] = [\phi_{n,m} \circ \gamma_m]\}$$

of the direct product, with the projections pr_n as the limit maps. Then

$$\Phi := (L^p(X, \mu, \phi_n))_{n \in \mathbb{N}} : L^p(X, \mu, E) \rightarrow \lim_{\leftarrow} L^p(X, \mu, E_n)$$

is a continuous linear map. If $\Phi(\gamma) = 0$, then there are subsets $A_n \subseteq X$ such that $\mu(A_n) = 0$ and $\phi_n \circ \gamma|_{x \setminus A_n} = 0$. After replacing $\gamma(x)$ with 0 for x in the set $\bigcup_{n \in \mathbb{N}} A_n$ of measure 0, we obtain $\gamma = 0$. Hence Φ is injective. To see

that Φ is surjective, let $([\gamma_n])_{n \in \mathbb{N}} \in \varprojlim L^p(X, \mu, E_n)$. For all $n \leq m$, there is a subset $A_{n,m} \subseteq X$ such that $\mu(A_{n,m}) = 0$ and

$$\phi_{n,m} \circ \gamma_m|_{A_{n,m}} = \gamma_n|_{A_{n,m}}.$$

Then the countable union $A := \bigcup_{n \in \mathbb{N}} \bigcup_{m \geq n} A_{n,m}$ has measure 0 as well and

$$\phi_{n,m} \circ \gamma_m|_A = \gamma_n|_A \quad \text{for all } n \leq m \text{ in } \mathbb{N}.$$

After re-defining $\gamma_n(x) := 0$ for $x \in A$, we may assume that $\gamma_n = \phi_{n,m} \circ \gamma_m$ for all $n, m \in \mathbb{N}$ with $n \leq m$. For each $x \in X$, we have

$$\gamma(x) := (\gamma_n(x))_{n \in \mathbb{N}} \in E;$$

thus $\phi_n(\gamma(x)) = \gamma_n(x)$ for all $n \in \mathbb{N}$. Note that $\prod_{n \in \mathbb{N}} \text{span}(\gamma_n(X))$ is a separable metrizable vector space and $\text{im}(\gamma)$ is contained in

$$F := E \cap \prod_{n \in \mathbb{N}} \text{span}(\gamma_n(X)),$$

which is separable. Let $\{x_1, x_2, \dots\} \subseteq F$ be a countable dense subset and $(q_j)_{j \in \mathbb{N}}$ be a sequence of seminorms $q_1 \leq q_2 \leq \dots$ on E defining its vector topology. Then countable set

$$\mathcal{E} := \{F \cap B_{1/i}^{q_j}(x_k) : i, j, k \in \mathbb{N}\}$$

of balls is a basis for the topology on F , whence the Borel σ -algebra is generated by \mathcal{E} , i.e., $\mathcal{B}(F) = \sigma(\mathcal{E})$. After increasing each of the q_j in turn, we may assume that $q_j = Q_j \circ \phi_{n_j}$ for some $n_j \in \mathbb{N}$ and some continuous seminorm Q_j on E_{n_j} . Since

$$\begin{aligned} \gamma^{-1}(B_{1/i}^{q_j}(x_k)) &= \{x \in X : q_j(\gamma(x) - x_k) < 1/i\} \\ &= \{x \in X : Q_j(\gamma_j(x) - \phi_j(x_k)) < 1/i\} = \gamma_j^{-1}(B_{1/i}^{Q_j}(\phi_j(x_k))) \end{aligned}$$

is measurable for all $i, j, k \in \mathbb{N}$, we deduce that γ is measurable. Since $\|\gamma\|_{\mathcal{L}^p, q_j} = \|\gamma_j\|_{\mathcal{L}^p, Q_j} < \infty$ for each $j \in \mathbb{N}$, we see that $\gamma \in \mathcal{L}^p(X, \mu, E)$. By construction, $\Phi([\gamma]) = ([\gamma_n])_{n \in \mathbb{N}}$. Thus Φ is surjective. To see that Φ is a topological embedding, note that seminorms of the form $\|\cdot\|_{L^p, Q \circ \phi_j}$ define the vector topology on $L^p(X, \mu, E)$, for $j \in \mathbb{N}$ and Q ranging through the

continuous seminorms on E_n . Since $\|\cdot\|_{L^p, Q} \circ \text{pr}_j$ is a continuous seminorm on $\varprojlim L^p(X, \mu, E)$ and

$$\|\cdot\|_{L^p, Q} \circ \text{pr}_j \circ \Phi = \|\cdot\|_{L^p, Q \circ \phi_j},$$

we see that Φ^{-1} is continuous and thus Φ a topological embedding. \square

Proof of Lemma 7.2. After passing to an isomorphic locally convex space, we may assume that E is the closed vector subspace

$$\varprojlim E_n = \{(x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} E_n : (\forall n \leq m) \phi_{n,m}(x_m) = x_n\}$$

of the direct product, with the projection onto the component E_n as the limit map ϕ_n . We realize $\varprojlim L^p(X, \mu, E_n)$ as the vector subspace

$$\{([\gamma_n])_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} L^p(X, \mu, E_n) : (\forall n \leq m) [\gamma_n] = [\phi_{n,m} \circ \gamma_m]\}$$

of the direct product, with the projections pr_n as the limit maps. Then

$$\Phi := (L^p(X, \mu, \phi_n))_{n \in \mathbb{N}} : L^p(X, \mu, E) \rightarrow \varprojlim L^p(X, \mu, E_n)$$

is a continuous linear map. Let $\gamma \in L^p(X, \mu, E)$ such that $\Phi(\gamma) = 0$. Using Lemma 1.11, we find a sequence $(q_j)_{j \in \mathbb{N}}$ of continuous seminorms q_j on E such that the $q_j|_F$ define a Hausdorff vector topology on $F := \overline{\text{span}(\gamma(X))}$. After increasing q_j if necessary, we may assume that $q_j = Q_j \circ \phi_{n_j}$ for some $n_j \in \mathbb{N}$ and some continuous seminorm Q_n on E_{n_j} . Since $\|\phi_{n_j} \circ \gamma\|_{L^\infty, Q_n} = 0$, there is a measurable set $A_n \subseteq X$ such that $\mu(A_n) = 0$ and $\phi_{n_j} \circ \gamma|_{X \setminus A_n} = 0$. After redefining $\gamma(x) := 0$ on the set $\bigcup_{n \in \mathbb{N}} A_n$ of measure 0, we achieve that $q_j(\gamma(x)) = 0$ for all $x \in X$ and $j \in \mathbb{N}$, whence $\gamma(x) = 0$ by choice of the q_j . Thus $[\gamma] = 0$ and thus Φ is injective.

To see that Φ is surjective, let $([\gamma_n])_{n \in \mathbb{N}} \in \varprojlim L_{rc}^\infty(X, \mu, E_n)$. As in the preceding proof, we may assume that $\gamma_n = \phi_{n,m} \circ \gamma_m$ for all $n, m \in \mathbb{N}$ with $n \leq m$. For each $x \in X$, we have

$$\gamma(x) := (\gamma_n(x))_{n \in \mathbb{N}} \in E;$$

thus $\phi_n(\gamma(x)) = \gamma_n(x)$ for all $n \in \mathbb{N}$. By Tychonoff's Theorem, the metrizable topological space $\prod_{n \in \mathbb{N}} \overline{\gamma_n(X)}$ is compact, whence also the closed subset

$$E \cap \prod_{n \in \mathbb{N}} \overline{\gamma_n(X)}$$

is metrizable and compact. Since $\text{im}(\gamma)$ is contained in the latter set, we deduce that $\overline{\text{im}(\gamma)}$ is compact and metrizable. If we can show that γ is measurable, the $\gamma \in \mathcal{L}_{rc}^\infty(X, E)$ and $\Phi([\gamma]) = ([\gamma_n])_{n \in \mathbb{N}}$ by construction, completing the proof of surjectivity. Using Lemma 1.11, we find a sequence $(q_j)_{j \in \mathbb{N}}$ of continuous seminorms $q_1 \leq q_2 \leq \dots$ on E such that the $q_j|_F$ define a separable Hausdorff vector topology \mathcal{O}' on $F := \overline{\text{span}(\text{im}(\gamma))}$. As the latter induced the given topology on the compact set $\overline{\gamma(X)}$, we need only show that γ is measurable as a mapping to (F, \mathcal{O}') . Let $\{x_1, x_2, \dots\}$ be a countable dense subset of (F, \mathcal{O}') . As in the preceding proof, we see that $\gamma: X \rightarrow (F, \mathcal{O}')$ is measurable.

To see that Φ is a topological embedding, note that seminorms of the form $\|\cdot\|_{L^\infty, Q \circ \phi_j}$ define the vector topology on $L_{rc}^\infty(X, \mu, E)$, for $j \in \mathbb{N}$ and Q ranging through the continuous seminorms on E_n . Since $\|\cdot\|_{L^\infty, Q} \circ \text{pr}_j$ is a continuous seminorm on $\varprojlim L_{rc}^\infty(X, \mu, E)$ and

$$\|\cdot\|_{L^\infty, Q} \circ \text{pr}_j \circ \Phi = \|\cdot\|_{L^\infty, Q \circ \phi_j},$$

we see that Φ^{-1} is continuous and thus Φ a topological embedding. \square

Proof of Lemma 7.3. Lemma 7.2 and its proof apply if we set $X := [a, b]$ and let μ be Lebesgue-Borel measure on X . Let $\Phi: L_{rc}^\infty([a, b], E) \rightarrow \varprojlim L_{rc}^\infty([0, 1], E_n)$ be as in the proof of Lemma 7.2. We can realize the projective limit $\varprojlim R([a, b], E_n)$ as a vector subspace of $\varprojlim L_{rc}^\infty([a, b], E_n)$. Then

$$\Phi_R := (R([a, b], \phi_n))_{n \in \mathbb{N}} = \Phi|_{R([a, b], E)}: R([a, b], E) \rightarrow \varprojlim R([a, b], E_n)$$

is a topological embedding (since Φ is a topological embedding). It only remains to see that Φ_R is surjective. If each E_n is a Fréchet space, then also E , $R([a, b], E)$ and each $R([a, b], E_n)$ are Fréchet spaces (cf. 1.33). Hence Φ_R has complete (and hence closed) image. As a consequence, we need only show that Φ_R has dense image in $\varprojlim R([a, b], E_n)$. Now $\text{im}(\Phi_R)$ is dense if and only if $R([a, b], \phi_j)(\text{im}(\Phi_R))$ is dense in $R([a, b], E_n)$ for each $n \in \mathbb{N}$. It suffices to show that $\text{pr}_n(\text{im}(\Phi_R)) = \text{im}(\text{pr}_n \circ \Phi_R) = \text{im } R([a, b], \phi_n)$ is dense in $R([a, b], E_n)$ for each $n \in \mathbb{N}$. This will hold if we can show that $\mathcal{T}([a, b], E) \subseteq \text{im } R([a, b], \phi_n)$. To do so, let $\gamma \in \mathcal{T}([a, b], E_n)$. Then $\text{im } \gamma = \{y_1, \dots, y_\ell\}$ with some $\ell \in \mathbb{N}$ and pairwise distinct elements $y_1, \dots, y_\ell \in E_n$. If $V \subseteq E_n$ is an open 0-neighbourhood, we find $z_j \in E$ such that $\phi_n(z_j) \in y_j + V$ for all $j \in \{1, \dots, \ell\}$, since $\text{im}(\phi_n)$ is dense in E_n by Lemma B.1. If we set $\eta(t) := z_j$ for all $t \in [a, b]$ such that $\gamma(t) = y_j$, we obtain a function $\eta \in$

$\mathcal{T}([a, b], E)$ such that $(\phi_n \circ \eta)(t) - \gamma(t) = \phi_n(z_j) - y_j \in V$ for all $t \in [a, b]$ (where $j \in \{1, \dots, \ell\}$ is chosen such that $\gamma(t) = y_j$). Hence $\gamma \in \overline{\text{im } R([a, b], \phi_n)}$, as desired. \square

Proof of Lemma 7.6. If G has a projective limit chart, let $\alpha_{n,m}: E_m \rightarrow E_n$, $\alpha_n: E \rightarrow E_n$, $\phi_n: U_n \rightarrow V_n \subseteq E_n$ and $\phi: U \rightarrow V \subseteq E$ be as in Definition 7.5. Let $x_n := q_n(e) \in V_n$; then

$$\beta_n := T_{x_n}(\phi_n)^{-1}: E_n \rightarrow L(G_n)$$

is an isomorphism of topological vector spaces, identifying E_n with $\{x_n\} \times E_n = T_{x_n}E_n$ via $(x_n, y) \mapsto y$. Therefore $W_n := \beta_n(V_n)$ is open in $L(G_n)$ and

$$\psi_n := \beta_n \circ \phi_n: U_n \rightarrow W_n$$

is a C^∞ -diffeomorphism. From

$$\phi_n \circ q_{n,m}|_{U_m} = \alpha_{n,m} \circ \phi_m,$$

we deduce that $(T_e \phi_n) \circ L(q_{n,m}) = (T_{x_m} \alpha_{n,m}) \circ (T_e \phi_m)$ and thus

$$L(q_{n,m}) \circ \beta_m = \beta_n \circ \alpha_{n,m},$$

entailing that

$$L(q_{n,m}) \circ \psi_m = L(q_{n,m}) \circ \beta_m \circ \phi_m = \beta_n \circ \alpha_{n,m} \circ \phi_m = \beta_n \circ \phi_n \circ q_{n,m}|_{U_m} = \psi_n \circ q_{n,m}|_{U_m}.$$

Thus (46) holds. Moreover,

$$L(q_{n,m})(W_m) = L(q_{n,m})(\beta_m(V_m)) = \beta_n(\alpha_{n,m}(V_m)) \subseteq \beta_n(V_n) = W_n.$$

Set $x := \phi(e) \in E$ and define

$$\beta := T_x \phi^{-1}: E \rightarrow L(G),$$

identifying $T_x E = \{x\} \times E$ with E . Then $W := \beta(V)$ is open in $L(G)$ and

$$\psi := \beta \circ \phi: U \rightarrow W$$

is a C^∞ -diffeomorphism. From $\alpha_n \circ \phi = \phi_n \circ q_n|_U$ we deduce that

$$\alpha_n \circ d\phi|_{L(G)} = d\phi_n|_{L(G_n)} \circ L(q_n)$$

and hence

$$\beta_n \circ \alpha_n = L(q_n) \circ \beta.$$

Thus

$$L(q_n)(W) = L(q_n)(\beta(V)) = \beta_n(\alpha_n(V)) \subseteq \beta_n(V_n) = W_n,$$

i.e., (47) holds. Moreover,

$$L(q_n) \circ \psi = L(q_n) \circ \beta \circ \phi = \beta_n \circ \alpha_n \circ \phi = \beta_n \circ \phi_n \circ q_n|_U = \psi_n \circ q_n|_U,$$

i.e., (48) holds. Since $W_n = \beta_n(V_n)$, we have

$$L(q_n)^{-1}(W_n) = (\beta_n^{-1} \circ L(q_n))^{-1}(V_n) = ((\alpha_n \circ \beta^{-1})^{-1}(V_n) = \beta(\alpha_n^{-1}(V_n))$$

and hence

$$\bigcap_{n \in \mathbb{N}} L(q_n)^{-1}(W_n) = \beta \left(\bigcap_{n \in \mathbb{N}} \alpha_n^{-1}(V_n) \right) = \beta(V) = W.$$

The proof of the converse implication is similar; given $\psi_n: U_n \rightarrow W_n \subseteq L(G_n)$ and $\psi: U \rightarrow W \subseteq L(G)$ as in Lemma 7.6, let E be the modelling space of G and E_n be the modelling space of E_n . Let $\beta: L(G) \rightarrow E$ and $\beta_n: L(G_n) \rightarrow E_n$ be an isomorphism of topological vector spaces. Then the conditions described in Definition 7.5 are satisfied if we set

$$V_n := \beta_n(W_n) \quad \text{and} \quad \alpha_n := \beta_n \circ L(q_n) \circ \beta^{-1}$$

for $n \in \mathbb{N}$,

$$\alpha_{n,m} := \beta_n \circ L(q_{n,m}) \circ \beta_m^{-1}$$

for positive integers $n \leq m$, and $V := \beta(W)$. □

C Details for Section 11

In this appendix, we prove Proposition 11.4 from Section 11 and discuss various concepts which are useful for the proof.

C.1 Recall that, if M is a C^k -manifold modelled on a locally convex space E with $k \in \mathbb{N} \cup \{\infty\}$ and $p \in M$, then $T_p M$ is the space of all tangent vectors

v to M at p . Interpreting these as geometric tangent vectors, they are \sim -equivalence classes $[\gamma]$ of C^k -curves $\gamma:]-\varepsilon, \varepsilon[\rightarrow M$ with $\gamma(0) = p$, where $\gamma \sim \eta$ if and only if

$$(\phi \circ \gamma)'(0) = (\phi \circ \eta)'(0) \quad (118)$$

for some (and hence every) chart ϕ of M around p . It is useful for us to deviate from this classical definition and consider, instead, (larger) equivalence classes $v = [\gamma]$ with continuous curves

$$\gamma: J \rightarrow M$$

defined on a non-degenerate interval $J \subseteq \mathbb{R}$ with $0 \in J$ such that $\gamma(0) = p$ and γ is *differentiable at 0* in the sense that $\phi \circ \gamma$ is differentiable at 0 for some (and hence every) chart of M around p (cf. Lemma 1.57). Also for such curves, we write $\gamma \sim \eta$ if and only if (118) holds for some (and hence every)³⁴ chart ϕ of M around p .

Definition C.2 Let M be a C^k -manifold modelled on a locally convex space E with $k \in \mathbb{N} \cup \{\infty\}$. Let $J \subseteq \mathbb{R}$ be a non-degenerate interval, $\eta: J \rightarrow M$ be a continuous curve and $t \in J$. We say that η is *differentiable at t* if the E -valued curve $\phi \circ \eta$ is differentiable at t for some (and hence every) chart ϕ of M around p . In this case, we set $\dot{\eta}(t) := [s \mapsto \eta(t + s)]$.

Lemma C.3 Let M be a C^k -manifold modelled on a Fréchet space E with $k \in \mathbb{N} \cup \{\infty\}$ and $\eta \in AC_{L^1}([a, b], M)$ with real numbers $a < b$. Write $\dot{\eta} = [\gamma]$ as in Definition 5.1. Then there is a Borel set $A \subseteq [a, b]$ with $\lambda_1(A) = 0$ such that η is differentiable at each $t \in [a, b] \setminus A$ and

$$\dot{\eta}(t) = \gamma(t)$$

for all $t \in [a, b] \setminus A$, with $\dot{\eta}(t)$ as in Definition C.2. The same conclusion holds if M is a C^k -manifold modelled on a strict (LF)-space which is a union $M = \bigcup_{n \in \mathbb{N}} M_n$ of C^k -manifolds $M_1 \subseteq M_2 \subseteq \dots$ modelled on Fréchet spaces such that the inclusion maps $M_n \rightarrow M_{n+1}$ and $j_n: M_n \rightarrow M$ are topological embeddings and C^k for all $n \in \mathbb{N}$, and $AC_{L^1}([a, b], M) = \bigcup_{n \in \mathbb{N}} AC_{L^1}([a, b], M_n)$ as a set.

³⁴See Lemma 1.57.

Proof. In the first situation, let $a = t_0 < t_1 < \dots < t_m = b$ such that $\phi_i([t_{i-1}, t_i]) \subseteq U_i$ for a chart $\phi_i: U_i \rightarrow V_i \subseteq E$, for each $i \in \{1, \dots, m\}$. Then $\eta_i := \phi_i \circ \eta|_{[t_{i-1}, t_i]} \in AC_{L^1}([t_{i-1}, t_i], E)$ for $i \in \{1, \dots, m\}$. Write $\eta'_i = [\gamma_i]$ with $\gamma_i \in \mathcal{L}^1([t_{i-1}, t_i], E)$. There is a Borel set $A_i \subseteq [t_{i-1}, t_i]$ such that $\eta'_i(t)$ exists for all $t \in [t_{i-1}, t_i] \setminus A_i$ and $\eta'_i(t) = \gamma_i(t)$ (see Lemma 1.28). There is a Borel set $B_i \subseteq [t_{i-1}, t_i[$ of measure 0 such that $\gamma(t) = T(\phi_i^{-1})(\eta_i(t), \gamma_i(t))$ for all $t \in [t_{i-1}, t_i[\setminus B_i$. Then $A := \{t_0, t_1, \dots, t_m\} \cup \bigcup_{i=1}^m (A_i \cup B_i)$ has the required properties.

In the second situation, let $\zeta \in AC_{L^1}([a, b], M)$. By hypothesis, we have $\zeta \in AC_{L^1}([a, b], M_n)$ for some $n \in \mathbb{N}$; write η for ζ , considered as a curve in M_n . Using that M_n carries the topology induced by M , for suitable $a = t_0 < t_1 < \dots < t_m = b$ we find charts $\phi_i: U_i \rightarrow V_i \subseteq E_n$ for M_n and $\psi_i: X_i \rightarrow Y_i \subseteq E$ for M such that $U_i \subseteq X_i$ and $\eta([t_{i-1}, t_i]) \subseteq U_i$ for all $i \in \{1, \dots, m\}$. Let η_i, γ_i, γ with $\dot{\eta} = [\gamma]$, A, A_i and B_i be as in the first situation. Write $\dot{\zeta} = [\theta]$. Set $\zeta_i := \psi_i \circ \zeta|_{[t_{i-1}, t_i]}$. Then $\zeta_i = \tau_i \circ \eta_i$ with $\tau_i := \psi_i \circ \phi_i^{-1}$ entails $\zeta'_i = [d\tau_i \circ (\eta_i, \gamma_i)]$, whence there is a Borel set $C_i \subseteq [t_{i-1}, t_i[$ of measure zero such that $\theta(t) = T\psi_i^{-1}(\zeta_i(t), d\tau_i(\eta_i(t), \gamma_i(t)))$ for all $t \in [t_{i-1}, t_i[\setminus C_i$ and thus

$$\theta(t) = T\psi_i^{-1}T\tau_i(\eta_i(t), \gamma_i(t)) = Tj_nT\phi_i^{-1}(\eta_i(t), \gamma_i(t)) = Tj_n\gamma(t)$$

for all $t \in [t_{i-1}, t_i[\setminus (B_i \cup C_i)$. Since $\zeta'_i(t) = d\tau_i(\eta_i(t), \eta'_i(t))$ for all $t \in]t_{i-1}, t_i[\setminus A_i$, we also deduce that $\zeta'(t)$ exists for all $t \in]t_{i-1}, t_i[\setminus A_i$, and is given by

$$\zeta'(t) = T\psi_i^{-1}(\zeta_i(t), \zeta'_i(t)) = T\psi_i^{-1}(\zeta_i(t), d\tau_i(\eta_i(t), \eta'_i(t))) = \theta(t)$$

for $t \in]t_{i-1}, t_i[\setminus (A_i \cup B_i \cup C_i)$. Summing up, $\zeta'(t)$ exists for all $t \in [a, b] \setminus (A \cup C_1 \cup \dots \cup C_m)$ and coincides with $\theta(t)$ there. \square

Let M be a σ -compact finite-dimensional smooth manifold. For $p \in M$, let $\varepsilon_p: \text{Diff}_c(M) \rightarrow M$ be the smooth map $\phi \mapsto \phi(p)$ (see, e.g., [22] for the smoothness). Let $C_c^\infty(M, TM)$ be the set of all smooth maps $X: M \rightarrow TM$ such that $\{p \in M: X(p) \neq 0_{\pi_{TM}(X(p))}\}$ is relatively compact in M . For $\phi \in \text{Diff}_c(M)$, let

$$\Gamma_\phi := \{X \in C_c^\infty(M, TM): \pi_{TM} \circ X = \phi\},$$

which is a vector space under pointwise operations. For $\phi \in \text{Diff}_c(M)$ and $[\gamma] \in T_\phi \text{Diff}_c(M)$, define

$$\alpha_\phi([\gamma]): M \rightarrow TM, \quad p \mapsto T\varepsilon_p([\gamma]) = [\varepsilon_p \circ \gamma],$$

i.e., $\alpha_\phi([\gamma]) = (T\varepsilon_p([\gamma]))_{p \in M}$. Given $\psi \in \text{Diff}_c(M)$, let us write

$$\rho_\psi: \text{Diff}_c(M) \rightarrow \text{Diff}_c(M), \quad \phi \mapsto \phi \circ \psi$$

for right translation with ψ .

Lemma C.4 *Let $\phi \in \text{Diff}_c(M)$.*

- (a) *For each $[\gamma] \in T_\phi \text{Diff}_c(M)$, we have that $\alpha_\phi([\gamma]) \in \Gamma_\phi$.*
- (b) *The map $\alpha_\phi: T_\phi \text{Diff}_c(M) \rightarrow \Gamma_\phi$ is an isomorphism of vector spaces.*
- (c) *Let $\psi \in \text{Diff}_c(M)$. For each $X \in \Gamma_\phi$, we have $X \circ \psi \in \Gamma_{\phi \circ \psi}$. The map*

$$R_\phi(\psi): \Gamma_\phi \rightarrow \Gamma_{\phi \circ \psi}, \quad X \mapsto X \circ \psi$$

is an isomorphism of vector spaces such that

$$R_\phi(\psi) \circ \alpha_\phi = \alpha_{\phi \circ \psi} \circ T_\phi \rho_\psi.$$

- (d) *α_{id_M} is the usual identification of $L(\text{Diff}_c(M)) = T_{\text{id}_M} \text{Diff}_c(M)$ with $\mathcal{X}_c(M)$, i.e., it coincides with $d\Phi|_{L(\text{Diff}_c(M))}$ where Φ is one of the usual charts for $\text{Diff}_c(M)$ with $\Phi^{-1}: X \mapsto \exp_g \circ X$ (cf. 11.1).*

Proof. It is clear that $R_\phi(\psi)$ takes Γ_ϕ into $\Gamma_{\phi \circ \psi}$, and is a linear map. Moreover, $R_\phi(\text{id}_M) = \text{id}_{\Gamma_\phi}$ and $R_{\phi \circ \psi}(\theta) \circ R_\phi(\psi) = R_\phi(\psi \circ \theta)$ if also $\theta \in \text{Diff}_c(M)$. We easily deduce that

$$R_\phi(\psi): \Gamma_\phi \rightarrow \Gamma_{\phi \circ \psi}$$

is an isomorphism of vector spaces with inverse $R_{\phi \circ \psi}(\psi^{-1})$. If $[\gamma] \in T_\phi \text{Diff}_c(M)$, then $T\varepsilon_p([\gamma]) \in T_{\varepsilon_p(\phi)}M = T_{\phi(p)}M$ and thus $\pi_{TM}(T\varepsilon_p([\gamma])) = \phi(p)$. Hence $\pi_{TM} \circ \alpha_\phi([\gamma]) = \phi$ and thus

$$\pi_{TM} \circ \alpha_\phi([\gamma]) = \phi. \tag{119}$$

As we do not know yet that the map $\alpha_\phi([\gamma]): M \rightarrow TM$ is smooth (nor compactly supported), we cannot conclude that $\alpha_\phi([\gamma]) \in \Gamma_\phi$ yet. Define

$$V_\phi := \text{im}(\alpha_\phi) \subseteq TM^M.$$

For $\psi \in \text{Diff}_c(M)$ and $[\gamma] \in T_\phi \text{Diff}_c(M)$, we have

$$\alpha_{\phi \circ \psi}(T\rho_\psi([\gamma])) = \alpha_{\phi \circ \psi}([t \mapsto \gamma(t) \circ \psi]) = (p \mapsto [t \mapsto \gamma(t)(\psi(p))]) = \alpha_\phi([\gamma])(\psi(p))$$

and thus

$$(\alpha_{\phi \circ \psi} \circ T\rho_\psi)([\gamma]) = \alpha_\phi([\gamma]) \circ \psi. \quad (120)$$

As a consequence,

$$f \circ \psi \in V_{\phi \circ \psi} \quad \text{for all } f \in V_\phi \text{ and } \psi \in \text{Diff}_c(M),$$

enabling us to define a map

$$r_\phi(\psi): V_\phi \rightarrow V_{\phi \circ \psi}, \quad f \mapsto f \circ \psi.$$

Note that $r_\phi(\text{id}_M) = \text{id}_{V_\phi}$ and $r_{\phi \circ \psi}(\theta) \circ r_\phi(\psi) = r_\phi(\psi \circ \theta)$ for each $\theta \in \text{Diff}_c(M)$. We easily deduce that $r_\phi(\psi)$ is an isomorphism of vector spaces for all $\phi, \psi \in \text{Diff}_c(M)$ (with inverse $r_{\phi \circ \psi}(\psi^{-1})$). It is clear that

$$W_\phi := \{f \in TM^M : \pi_{TM} \circ f = \phi\}$$

is a vector space under the pointwise operations. Since each of the maps $T_p \varepsilon_p: T_\phi \text{Diff}_c(M) \rightarrow T_{\phi(p)}M$ is linear, we deduce that α_ϕ is linear as a map to W_ϕ . In particular, V_ϕ is a vector subspace of W_ϕ and we can consider α_ϕ as a *linear* surjective map

$$\alpha_\phi: T_\phi \text{Diff}_c(M) \rightarrow V_\phi.$$

Fix a smooth Riemannian metric g on M , with Riemannian exponential function $\exp: \mathcal{D} \rightarrow M$ on an open neighbourhood $\mathcal{D} \subseteq TM$ of the zero-section. Consider the map

$$h: \mathcal{X}_c(M) \rightarrow T_{\text{id}_M} \text{Diff}_c(M), \quad X \mapsto [t \mapsto \exp \circ (tX)].$$

Then $h = T\Phi^{-1}(0, \bullet)$ for a typical chart Φ of $\text{Diff}_c(M)$ around id_M such that $\Phi(\text{id}_M) = 0$, with $\Phi^{-1}: X \mapsto \exp \circ X$. Now

$$(\alpha_{\text{id}_M} \circ h)(X) = ([t \mapsto \exp(tX(p))])_{p \in M} = X,$$

using that $c: t \mapsto \exp(tX(p))$ is the geodesic starting (for $t = 0$) at p with velocity $[t \mapsto c(t)] = \dot{c}(0) = X(p)$. Since h is an isomorphism of topological vector space, it is surjective and hence the injectivity of

$$X \mapsto (\alpha_{\text{id}_M} \circ h)(X) = X \quad (121)$$

entails that α_{id_M} is injective. Moreover, since h is surjective, we deduce from (121) that

$$V_{\text{id}_M} = \text{im}(\alpha_{\text{id}_M}) = \text{im}(\alpha_{\text{id}_M} \circ h) = \mathcal{X}_c(M) = \Gamma_{\text{id}_M}.$$

Thus

$$\alpha_{\text{id}_M} : T_{\text{id}_M} \text{Diff}_c(M) \rightarrow \Gamma_{\text{id}_M}$$

is an isomorphism of vector spaces and $\alpha_{\text{id}_M} = h^{-1} = d\Phi|_{L(\text{Diff}_c(M))}$, establishing (d). Now

$$\begin{aligned} V_\phi = \text{im}(\alpha_\phi) &= \text{im}(\alpha_\phi \circ T_{\text{id}_M}(\rho_\phi)) = \text{im}(r_{\text{id}_M}(\phi) \circ \alpha_{\text{id}_M}) \\ &= \text{im}(R_{\text{id}_M}(\phi) \circ \alpha_{\text{id}_M}) = \text{im}(R_{\text{id}_M}(\phi)) = \Gamma_\phi. \end{aligned}$$

Notably, $\text{im}(\alpha_\phi) = V_\phi \subseteq \Gamma_\phi$ and so (a) holds. Further, $\alpha_\phi : T_\phi \text{Diff}_c(M) \rightarrow V_\phi = \Gamma_\phi$ is an isomorphism, establishing (b). By (120), we have (c). \square

Remark C.5 By Lemma C.4 (b), we can identify $T_\phi \text{Diff}_c(M)$ with Γ_ϕ by means of the isomorphism α_ϕ (and we can give Γ_ϕ the unique locally convex vector topology making α_ϕ an isomorphism of topological vector spaces). By Lemma C.4 (c), the tangent map $T_\phi \rho_\psi$ of right translation with ψ on $\text{Diff}_c(M)$ corresponds to the right translation $R_\phi(\psi) : \Gamma_\phi \rightarrow \Gamma_{\phi \circ \psi}$.

Proof of Proposition 11.4. Write $M = \bigcup_{j \in \mathbb{N}} K_j$ with compact subsets $K_j \subseteq M$ such that K_j is contained in the interior of K_{j+1} , for each $j \in \mathbb{N}$. We know that the Fréchet-Lie group $\text{Diff}_{K_j}(M)$ is a submanifold of $\text{Diff}_c(M)$ for each $j \in \mathbb{N}$, and

$$\text{Diff}_c(M) = \bigcup_{j \in \mathbb{N}} \text{Diff}_{K_j}(M). \quad (122)$$

If $\eta \in AC_{L^1}([0, 1], \text{Diff}_c(M))$ and $\eta' = [\theta]$, then $\eta([0, 1])$ is a compact subset of $\text{Diff}_c(M)$ and hence contained in $\text{Diff}_{K_j}(M)$ for some $j \in \mathbb{N}$, by the compact regularity of the union (122) (which follows from [31, Corollary 3.6] and [28, Remark 5.2]). Hence $\eta \in AC_{L^1}([0, 1], \text{Diff}_{K_j}(M))$ (see Lemma 4.12 and Remark 4.14). By Lemma C.3, we find a Borel subset $B \subseteq [0, 1]$ of measure $\lambda_1(B) = 0$ such that η is differentiable at each $t \in [0, 1] \setminus B$, with $\dot{\eta}(t) = \theta(t)$. Assume that $\eta(0) = \text{id}_M$.

If $\eta = \text{Evol}^r([\gamma])$, i.e., η is a Carathéodory solution to

$$y'(t) = \gamma(t).y(t)$$

(with multiplication in the tangent Lie group $T \operatorname{Diff}_c(M)$), then

$$\dot{\eta}(t) = \gamma(t) \cdot \eta(t)$$

for all $t \in [0, 1] \setminus A$, after increasing A if necessary. Note that we identify $L(\operatorname{Diff}_c(M))$ with $\mathcal{X}_c(M)$ in Proposition 11.4 by means of $\alpha_{\operatorname{id}_M}$ (with notation as in Lemma C.4. Making this identification, $\alpha_{\operatorname{id}_M}$ turns into an identity map. Now

$$([s \mapsto \eta(t+s)(p)])_{p \in M} = \alpha_{\eta(t)}(\dot{\eta}(t)) = \alpha_{\operatorname{id}_M}(\gamma(t)) \circ \eta(t) = \gamma(t) \circ \eta(t) \quad (123)$$

for each $t \in [0, 1] \setminus A$, exploiting Lemma C.4 (c). For each $p \in M$, we have $\eta_p := \varepsilon_p \circ \eta \in AC_{L^1}([0, 1], M)$. Moreover, $\eta'_p(t) = (\varepsilon_p \circ \eta)'(t)$ exists for all $t \in [0, 1] \setminus A$ and is given by

$$\eta'_p(t) = (\varepsilon_p \circ \eta)'(t) = [s \mapsto \eta(t+s)(p)] = \gamma(t)(\eta(t)(p)) = \gamma(t)(\eta_p(t))$$

(see (123)). Hence η_p is a Carathéodory solution to

$$y'(t) = \gamma(t)(y(t)) = f(t, y(t)), \quad y(0) = p.$$

Thus f admits a global flow for initial time $t_0 = 0$ and $\eta(t)(p) = \eta_p(t) = \Phi_{t,0}^f(p)$ for all $p \in M$ and $t \in [0, 1]$, i.e., $\eta(t) = \Phi_{t,0}^f$.

Conversely, assume that f admits a global flow for initial time $t_0 := 0$ and $\eta(t) = \Phi_{t,t_0}^f$ for each $t \in [0, 1]$. As we assume that the finite-dimensional smooth manifold M is σ -compact, there is a countable dense subset $D \subseteq M$. For each $p \in D$, the curve

$$\eta_p := \varepsilon \circ \eta \in AC_{L^1}([0, 1], M)$$

is a Carathéodory solution to

$$y'(t) = \gamma(t)(y(t)) = f(t, y(t)), \quad y(0) = p.$$

Hence, there is a Borel set $A_p \subseteq [0, 1]$ with $\lambda_1(D_p) = 0$ such that η_p is differentiable at each $t \in [0, 1] \setminus A_p$ and

$$\eta'_p(t) = f(t, \eta_p(t)) = \gamma(t)(\eta_p(t)). \quad (124)$$

Now $A := B \cup \bigcup_{p \in D} A_p$ is a Borel set with $\lambda_1(A) = 0$. For each $t \in [0, 1] \setminus A$, we know that η is differentiable at t . Thus

$$\alpha_{\eta(t)}(\eta'(t)) = ([t \mapsto \eta(t)(p)])_{p \in M} \in \Gamma_{\eta(t)}$$

can be formed and is a smooth (and hence continuous) function $M \rightarrow TM$. Also $\gamma(t) \circ \eta(t): M \rightarrow TM$ is continuous. Therefore

$$\alpha_{\eta(t)}(\dot{\eta}(t)) = \gamma(t) \circ \eta(t) \quad (125)$$

will hold if we can show that

$$(\alpha_{\eta(t)}(\eta'(t)))(p) = \gamma(t)(\eta(t)(p))$$

for all $p \in D$, which can be rewritten as

$$\dot{\eta}_p(t) = \gamma(\eta_p(t)), \quad (126)$$

noting that $[t \mapsto \eta(t)(p)] = [t \mapsto \eta_p(t)] = \dot{\eta}_p(t)$. Since (126) holds by (124), we have established (125). Using Lemma C.4 (c), we can rewrite (125) as

$$\dot{\eta}(t) = \gamma(t) \cdot \eta(t).$$

Thus $\theta(t) = \dot{\eta}(t) = \gamma(t) \cdot \eta(t)$ for all $[0, 1] \setminus A$, entailing that η is a Carathéodory solution to $y'(t) = f(t, y(t))$, $y(0) = \text{id}_M$ and thus $\eta = \text{Evol}^r([\gamma])$. \square

References

- [1] Alzaareer, H., *Lie groups of mappings on non-compact spaces and manifolds*, Doctoral Dissertation, Universität Paderborn, 2013; see <http://nbn-resolving.de/urn:nbn:de:hbz:466:2-11572>
- [2] Alzaareer, H. and A. Schmeding, *Differentiable mappings on products with different degrees of differentiability in the two factors*, Expo. Math. **33** (2015), 184–222.
- [3] Bauer, H., “Maß- und Integrationstheorie,” de Gruyter, Berlin, ²1992.
- [4] Berg, C., J.P.R. Christensen and P. Ressel, “Harmonic Analysis on Semi-groups,” Springer, New York, 1984.
- [5] Bertram, W., H. Glöckner and K.-H. Neeb, *Differential calculus over general base fields and rings*, Expo. Math. **22** (2004), 213–282.
- [6] Biller, H., *Analyticity and naturality of the multi-variable functional calculus*, Expo. Math. **25** (2007), no. 2, 131–163.
- [7] Bochnak, J. and J. Siciak, *Analytic functions in topological vector spaces*, Studia Math. **39** (1971), 77–112.

- [8] Bonet, J., S. Dierolf, and C. Fernández, *Inductive limits of vector-valued sequence spaces*, Publ. Mat., Barc. **33** (1989), no. 2, 363–367.
- [9] Bourbaki, N., “Topological Vector Spaces, Chapters 1–5,” Springer, 1987.
- [10] Bourbaki, N., “Lie Groups and Lie Algebras,” Chapters 1–3, Springer, 1989.
- [11] Bruhat, F. and H. Whitney, *Quelques propriétés fondamentales des ensembles analytiques-réels*, Comment. Math. Helv. **33** (1959), 132–160.
- [12] Dahmen, R., *Analytic mappings between LB-spaces and applications in infinite-dimensional Lie theory*, Math. Z. **266** (2010), 115–140.
- [13] Dahmen, R., “Direct Limit Constructions in Infinite-Dimensional Lie Theory,” Doctoral Dissertation, Universität Paderborn, May 2011; see [urn:nbn:de:hbz:466:2-239](#)
- [14] Dahmen, R., *Regularity in Milnor’s sense for ascending unions of Banach-Lie groups*, J. Lie Theory **24** (2014), 545–560.
- [15] Dahmen, R. and H. Glöckner, *Bounded solutions of finite lifetime to differential equations in Banach spaces*, Acta Sci. Math. (Szeged) **81** (2015), 457–468.
- [16] Dahmen, R., H. Glöckner, and A. Schmeding, *Complexifications of infinite-dimensional manifolds and new constructions of infinite-dimensional Lie groups*, preprint, [arXiv:1410.6468](#).
- [17] Engelking, R., “General Topology,” Heldermann-Verlag, Berlin, 1989.
- [18] Florencia, M., F. Mayoral, and P. J. Paúl, *Inductive limits of spaces of vector-valued integrable functions*, Results Math. **25** (1994), no. 3–4, 242–251.
- [19] Glöckner, H., *Lie groups without completeness restrictions*, Banach Center Publ. **55** (2002), 43–59.
- [20] Glöckner, H., *Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups*, J. Funct. Anal. **194** (2002), 347–409.
- [21] Glöckner, H., *Algebras whose groups of units are Lie groups*, Studia Math. **153** (2002), 147–177.
- [22] Glöckner, H., *Patched locally convex spaces, almost local mappings, and the diffeomorphism groups of non-compact manifolds*, manuscript, 2002.
- [23] Glöckner, H., *Direct limit Lie groups and manifolds*, J. Math. Kyoto Univ. **43** (2003), 1–26.
- [24] Glöckner, H., *Lie groups of measurable mappings*, Canad. J. Math. **55** (2003), 969–999.

- [25] Glöckner, H., *Lie groups over non-discrete topological fields*, preprint, [arXiv:math/0408008](#).
- [26] Glöckner, H., *Diff(\mathbb{R}^n) as a Milnor-Lie group*, Math. Nachr. **278** (2005), 1025–1032.
- [27] Glöckner, H., *Fundamentals of direct limit Lie theory*, Compositio Math. **141** (2005), 1551–1577.
- [28] Glöckner, H., *Direct limits of infinite-dimensional Lie groups compared to direct limits in related categories*, J. Funct. Anal. **245** (2007), 19–61.
- [29] Glöckner, H., *Direct limits of infinite-dimensional Lie groups*, pp. 243–280 in: K.-H. Neeb and A. Pianzola (Eds.), “Developments and Trends in Infinite-Dimensional Lie Theory,” Progr. Math. **288**, Birkhäuser, Boston, 2011.
- [30] Glöckner, H., *Regularity properties of infinite-dimensional Lie groups*, Oberwolfach Rep. **13** (2013), 791–794.
- [31] Glöckner, H., *Homotopy groups of ascending unions of infinite-dimensional manifolds*, to appear in Ann. Inst. Fourier (Grenoble); cf. [arXiv:0812.4713](#).
- [32] Glöckner, H., *Finite order differentiability properties, fixed points and implicit functions over valued fields*, preprint, [arXiv:math/0511218](#).
- [33] Glöckner, H., *Implicit functions from topological vector spaces to Fréchet spaces in the presence of metric estimates*, preprint, [arXiv:math/0612673](#).
- [34] Glöckner, H., *Regularity properties of infinite-dimensional Lie groups, and semiregularity*, preprint, [arXiv:1208.0715](#).
- [35] Glöckner, H., *Differentiable mappings between spaces of sections*, preprint, [arXiv:1308.1172](#).
- [36] Glöckner, H., *Fundamentals of submersions and immersions between infinite-dimensional manifolds*, [arXiv:1502.05795](#).
- [37] Glöckner, H. and K.-H. Neeb, *When unit groups of continuous inverse algebras are regular Lie groups*, Studia Math. **211** (2012), 95–109.
- [38] Glöckner, H. and K.-H. Neeb, “Infinite-dimensional Lie Groups,” book in preparation.
- [39] Hamilton, R., *The inverse function theorem of Nash and Moser*, Bull. Am. Math. Soc. **7** (1982), 65–222.
- [40] Hervé, M., “Analyticity in Infinite-Dimensional Spaces,” de Gruyter, 1989.
- [41] Hille, E. and R. S. Phillips, “Functional Analysis and Semi-Groups,” AMS, Providence, 1957.

- [42] Jurdjevic, V., “Geometric Control Theory,” Cambridge University Press, 1997.
- [43] Kelley, J. L., “General Topology,” Springer, New York; reprint of 1955 ed.
- [44] Klose, D. and F. Schuricht, *Parameter dependence for a class of ordinary differential equations with measurable right-hand side*, Math. Nachr. **284** (2011), no. 4, 507–517.
- [45] Kriegl, A. and P. W. Michor, “The Convenient Setting of Global Analysis,” AMS, Providence, 1997,
- [46] Kriegl, A. and P. W. Michor, *Regular infinite-dimensional Lie groups*, J. Lie Theory **7** (1997), 61–99.
- [47] Lang, S., “Fundamentals of Differential Geometry”, Springer, 1999.
- [48] Maise, R. and D. Vogt, “Introduction to Functional Analysis,” Oxford Graduate Texts in Mathematics, Clarendon Press, Oxford, 1997.
- [49] Melikhov, S. N., *Absolutely convergent series in the canonical inductive limits*, Math. Notes **39** (1986), 475–480; translation from Mat. Zametki **39** (1986), no. 6, 877–886.
- [50] Michael, E. A., *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. **11** (1952), 79 pp.
- [51] Michor, P. W., “Manifolds of Differentiable Mappings,” Shiva Publ., Orpington, 1980.
- [52] Milnor, J., *Remarks on infinite-dimensional Lie groups*, pp. 1007–1057 in: B. S. DeWitt and R. Stora (eds.), “Relativité, groupes et topologie II,” North-Holland, Amsterdam, 1984.
- [53] Mujica, J., *Spaces of continuous functions with values in an inductive limit*, pp. 359–367 in: Zapata, G. I. (ed.), “Functional Analysis, Holomorphy, and Approximation Theory” (Proceedings, Rio de Janeiro, 1979), Lecture Notes Pure Appl. Math. **83**, Dekker, New York, 1983.
- [54] Neeb, K.-H., *Towards a Lie theory of locally convex groups*, Jpn. J. Math. **1** (2006), 291–468.
- [55] Neeb, K.-H. and H. Salmasian, *Differentiable vectors and unitary representations of Fréchet-Lie supergroups*, Math. Z. **275** (2013), no. 1-2, 419–451.
- [56] Neeb, K.-H. and F. Wagemann, *Lie group structures on groups of smooth and holomorphic maps on non-compact manifolds*, Geom. Dedicata **134** (2008), 17–60.
- [57] Ostling, E. G. and A. Wilansky, *Locally convex topologies and the convex compactness property*, Proc. Cambridge Philos. Soc. **75** (1974), 45–50.

- [58] Omori, H., Y. Maeda, A. Yoshioka and O. Kobayashi, *On regular Fréchet–Lie groups IV*, Tokyo J. Math. **5** (1982), 365–398.
- [59] Omori, H., Y. Maeda, A. Yoshioka and O. Kobayashi, *On regular Fréchet–Lie groups V*, Tokyo J. Math. **6** (1983), 39–64.
- [60] Rudin, W., “Real and Complex Analysis,” McGraw-Hill, 3rd ed., 1987.
- [61] Rudin, W., “Functional Analysis,” McGraw-Hill, 2nd ed., New York, 1991.
- [62] Schechter, E., “Handbook of Analysis and its Foundations,” Academic Press, 1997.
- [63] Schmeding, A., *The diffeomorphism group of a non-compact orbifold*, Diss. Math. **507**, 2015.
- [64] Schmets, J., *Spaces of vector-valued continuous functions*, pp.368–377 in: R. M. Aron and S. Dineen (eds.), “Vector Space Measures and Applications I” (Proceedings, Dublin, 1977), Lecture Notes in Math. **644**, Springer, 1978.
- [65] Schütt, J., *Symmetry groups of principal bundles over non-compact bases*, preprint, [arXiv:1310.8538](https://arxiv.org/abs/1310.8538).
- [66] Schubert, H., “Topologie,” B.G. Teubner, Stuttgart, 1964.
- [67] Schuricht, F. and H. von der Mosel, *Ordinary differential equations with measurable right-hand side and parameters in metric spaces*, Preprint 676, SFB 256, Bonn 2000; <http://www.math.tu-dresden.de/~schur/Dateien/forschung/papers/2000schurmoode.pdf>
- [68] Sontag, E.D., “Mathematical Control Theory: Deterministic finite-dimensional systems,” Springer, New York, ²1998.
- [69] Turpin, Ph., *Une remarque sur les algèbres à inverse continu*, C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A1686–A1689.
- [70] Voigt, J., *On the convex compactness property for the strong operator topology*, Note Mat. **12** (1992), 259–269.
- [71] von Weizsäcker, H., *In which spaces is every curve Lebesgue–Pettis integrable?*, preprint, [arXiv:1207.6034](https://arxiv.org/abs/1207.6034).
- [72] Waelbroeck, L., *Les algèbres à inverse continu*, C. R. Acad. Sci. Paris **238** (1954), 640–641.
- [73] Wockel, C., *Lie group structures on symmetry groups of principal bundles*, J. Funct. Anal. **251** (2007), 254–288.

Helge Glöckner, Universität Paderborn, Institut für Mathematik,
 Warburger Str. 100, 33098 Paderborn, Germany. Email: glockner@math.upb.de